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ORIGINAL ARTICLE

Solving space-fractional Cauchy problem by modified finite-difference discretization scheme

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Abstract This paper deals with the stability convergence analysis for SFCE in the sense of Riemann-Liouville derivative. A modified FDDS is developed utilizing the fractionally-shifted Grünwald formula in handling the SFCE. In this orientation, a novel operational matrix based on the implicit scheme is proposed for solving such issue. The stability features of steady states of the SFCE are investigated numerically. Several numerical applications using the well-known SFCE are tested to demonstrate the capability and feasibility of the method. The acquired results indicate that the proposed method is an appropriate tool for solving various fractional systems arises in physics and engineering.

1. Introduction

FDEs provide a crucial tool for modelling numerous phenomena that potentially arise, with various applications across physics, applied sciences and engineering. It is the classic calculus generalization that deals with operators of differentiation and integration of non-integer orders. They are considered as useful and powerful tools in both theoretical and practical fields to handle the complexity and nonlinearity of realistic issues, including non-Brownian motion, fluid flows, chemical science, management theory, signal process, fiber optics, systems identification, elastic materials, polymers, and so on [1–7]. Also, they are used to describing the memory and hereditary effort for multiple kinds of materials. Consequently, considerable attention has been given to handle FDEs. In specific, the PDEs of fractional-order are progressively are delineate in finance, viscoelasticity, mathematical biology, traffic, population and particle chemistry [8–11]. Totally different types of fractional PDEs have been recently studied and discussed by researchers. For example, but not limited to, the fractional diffusion equation [12], fractional advection-dispersion equation [13], space-time fractional diffusion-wave equation [14], fractional convection-diffusion equation [15], fractional heat and wave-like equations [16], fractional Korteweg-de Vries equation [17], and so forth [18–23].

Generally, it is not easy to find accurate solutions to this type of equation due to the presence of fractional derivatives

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in the mathematical systems created. Therefore, computer technologies are used to obtain acceptable approximate solutions in the field of study and then attempt to reach engineering and physical interpretations that enable us to gain a greater understanding of the natural phenomena of fractional order. Anyhow, there is no clear and explicit engineering or physical interpretation of fractional derivatives of different types similar to classical derivatives that lead us to a direct concept and precise meaning [6,24–29]. Thus, several powerful strategies are established and developed to interduce numerical-analytical solutions of such FDEs, including FDDS [30], finite volume technique [31], finite element technique [32], homotopy perturbation technique [33], residual power series technique [34–39], differential transform method [40–42], and reproducing kernel method [43–51].

During this paper, we tend to propose the FDDS to get approximate solution for the SFCP utilizing the Riemann-Liouville sense. More specifically, we consider the following form

$$\frac{\partial u(x,t)}{\partial t}+\epsilon\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}}=f(x), \tag{1}$$

subject to the initial condition

$$u(x,0)=g(x), \tag{2}$$

where ϵ is a positive parameter, $t>0, x\in[a,b]$, $f(x)$ is given source term, $g(x)$ is analytical smooth function of spatial x, $u(x,t)$ is unknown analytical function to be determined, and β is describing the order of space-fractional derivative such that $0<\beta\leq 1$. Here, the space-fractional derivative is considered under the Riemann-Liouville sense. Further, we assume that the IVPs (1) and (2) has unique analytical solution in term of spatial x and temporal t in the domain of interest.

The starting point for the FDDS is a partition of the computational domain $[a,b]$ into a finite number of sub-domains $V_i, i=0,1,2,\cdots,N$, known as control volumes, the union of all control volumes ought to cover the whole domain. However, we will use the fractionally-shifted Grünwald formula for the discretization of the space-fractional derivative under the concept of Riemann-Liouville fractional derivative, then an implicit scheme will be derived in order to solve IVPs (1) and (2) using fitted operational matrix form.

Furthermore, several scholars work on a fractional PDEs have been proposed. In [52], the domain decomposition method has been developed to approximate the solution for fractional convection-diffusion equation with a nonlinear source term. While, in [53], numerical solution for a class of fractional convection-diffusion equations has been introduced using the flatlet oblique multi-wavelets. The sinc–legendre collocation technique has been implemented for a category of fractional convection-diffusion equations with variable coefficients in [54]. In addition, Meerschaert and Tadjeran proposed the FDDS for resolution a class of fractional advection–dispersion flow equations [55]. The finite volume technique has been applied in solving fractional diffusion equations [56]. Later, the authors in [12] extend the finite volume method to two-dimensional fractional diffusion equation. In [13], the finite volume and FDDS have been utilized in handling the space-fractional advection-dispersion equation as well as the stability and convergence analysis of the scheme have been verified with accuracy is $\mathcal{O}(\tau+\mathrm{h})$ based on the use of fractionally-shifted Grünwald formula. For more specific details, the reader can be refers to [57-61].

This paper introduces an implicit scheme of the finite-difference in solving the SFCP with and without the source term based on the fractional Gronwald-Litnikov formula. After giving some preliminary definitions related to fractional calculus, the solution procedure of a modified finite-difference technique is presented in Section 3. Also, we discuss the stability and consistency of the proposed scheme. In Section 4, the numerical experiments are given to show the reliability and efficiency of the method. At last, a brief conclusion is outlined in the last section.

2. Preliminaries

This section is dedicated to providing some necessary definitions and mathematical preliminaries for fractional calculus, especially those related to the Grünwald-Letnikov fractional operator. Anyhow, when using a numerical technique to deal with a differential equation, it must be ensured that the resulting numerical solution is a sufficiently good approximation to the actual solution. Therefore, some necessary definitions and notes are presented in this section to discuss stability analysis.

Definition 1. The Riemann-Liouville integral of fractional order $\alpha>0$, $\mathcal{J}_a^{\alpha}u(x)$ is defined as

$$\mathcal{J}_a^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}u(t)dt, \tag{3}$$

Provided that $u\in L_1[a,b]$. For $\alpha=0$, we have $\mathcal{J}_a^0u(x)=u(x)$ the identity operator.

Definition 2. Let n be the smallest integer that exceeds α, then the Riemann-Liouville fractional derivative of order $\alpha>0$ is defined as

$$\begin{aligned}\mathscr{D}_a^{\alpha}u(x)&=\mathscr{D}^n\mathcal{J}_a^{(n-\alpha)}u(x)\\&=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d^n}{dx^n}\right)\left[\int_a^x\frac{u(t)}{(x-t)^{\alpha+1-n}}dt\right].\end{aligned} \tag{4}$$

Remembering that, for $\alpha=0$, we have $\mathscr{D}_a^0u(x)=u(x)$ is the identity operator and for $\alpha\in\mathbb{N}$, we have $\mathscr{D}_a^{\alpha}u(x)=\frac{d^{\alpha}u(x)}{dx^{\alpha}}$.

Definition 3:. If $\alpha>0, \mathrm{u}\in\mathrm{C}^{\lceil\alpha\rceil}[\mathrm{a,b}]$, then,

$$\tilde{\mathscr{D}}_a^{a}u(x)=\lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\left[\frac{x-a}{h}\right]}(-1)^k\binom{\alpha}{k}u(x-kh), a<x\leq b, \tag{5}$$

where $h=\frac{x-a}{N}$ is called the Grünwald-Letnikov fractional derivative of order α of the function $u(x)$.

The following theorem shows the relation between the Grünwald-Letnikov fractional derivatives and Riemann- Liouville fractional derivatives.

Theorem 1. *Let* $\alpha>0, \mathrm{n}=\lceil\alpha\rceil$, *and* $\mathrm{u}\in\mathrm{C^n}[\mathrm{a,b}]$. *Then,*

$$\tilde{\mathscr{D}}_a^{\alpha}u(x)=\tilde{\mathscr{D}}_a^{\alpha}u(x), a<x\leq b. \tag{6}$$

Theorem 2. *Let* $\alpha > 0$ *and* $\mathrm{u} \in \mathrm{C}[\mathrm{a}, \mathrm{b}]$. *Then,*

$$J_a^\alpha u(x) = \frac{\lim}{h \to 0} h^\alpha \sum_{k=0}^{\left[\frac{x-a}{h}\right]} (-1)^k \binom{-\alpha}{k} u(x - kh), \tag{7}$$

where $h = \frac{x-a}{N}$, $a < x \le b, (-1)^k \binom{-\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k+1)}$, *and* $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$.

If we define weights $w_0^\alpha = 1$, $w_1^\alpha = \alpha$, and $w_k^\alpha = \left(1 - \frac{(1-\alpha)}{k}\right) w_{k-1}^\alpha, k = 2, 3, \cdots$, then we may rewrite (7) as:

$$J_a^\alpha u(x) = \lim_{h \to 0} h^\alpha \sum_{k=0}^{\left[\frac{x-a}{h}\right]} w_k^\alpha u(x - kh), h = \frac{x-a}{N}. \tag{8}$$

In fact, this formula is used to approximations the fractional integrals $J_a^\alpha u(x)$.

Lemma 1. *If* $0 < \alpha < 1$, *then one can write*

1. $w_0^\alpha = 1$ *and* $w_j^\alpha > 0$ *for* $j = 1, 2, \cdots$
2. $w_j^\alpha - w_{j+1}^\alpha > 0$ *for* $j = 0, 1, \cdots$.
3. $\lim_{j \to \infty} w_j^\alpha = 0$.

Proof. For the first part, using the induction so that $\mathrm{w}_0^\alpha = 1$ and $\mathrm{w}_1^\alpha = \alpha > 0$, thus from the recursive relation

$$w_j^\alpha = \left(1 - \frac{(1-\alpha)}{j}\right) w_{j-1}^\alpha, k = 2, 3, \cdots, \tag{9}$$

and since $0 < \alpha < 1$, we have $0 < \frac{1-\alpha}{j} < \frac{1}{j} < 1$ for $j \ge 2$. So, the coefficient $\left(1 - \frac{(1-\alpha)}{k}\right)$ in (9) is strictly between zero and one. For the second part, for $j \ge 2$, we have

$$\begin{aligned} w_j^\alpha - w_{j+1}^\alpha &= \left(1 - \frac{(1-\alpha)}{j}\right) w_{j-1}^\alpha - \left(1 - \frac{(1-\alpha)}{j+1}\right) w_j^\alpha \\ &= \left(1 - \frac{(1-\alpha)}{j}\right) w_{j-1}^\alpha - \left(1 - \frac{(1-\alpha)}{j+1}\right)\left(1 - \frac{(1-\alpha)}{j}\right) w_{j-1}^\alpha \\ &= \left[1 - \left(1 - \frac{(1-\alpha)}{j+1}\right)\right]\left(1 - \frac{(1-\alpha)}{j}\right) w_{j-1}^\alpha \\ &= \frac{(1-\alpha)}{j+1}\left(1 - \frac{(1-\alpha)}{j}\right) w_{j-1}^\alpha \end{aligned} \tag{10}$$

Hence, $0 < w_{j+1}^\alpha < w_j^\alpha < w_1^\alpha = \alpha < 1 = w_0^\alpha$ for $j \ge 2$. So, $\lim_{j \to \infty} w_j^\alpha = 0$.

Now, to analyze the stability of the difference scheme for Cauchy problem, let v be a vector in ℓ_2 such that $v = (\cdots, v_{-1}, v_0, v_1, \cdots)^T$ and then define the discrete Fourier transform of v.

Definition 4. The discrete Fourier transform of $\mathrm{v} \in \ell_2$ is the operate $\widehat{\mathrm{v}} \in \mathrm{L}_2[-\pi, \pi]$ defined by

$$\widehat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-i\xi} v_m \ \text{for} \xi \in [-\pi, \pi]. \tag{11}$$

Definition 5. The symbol of difference scheme $v^{n+1} = Qv^n$ is the coefficient of $\widehat{v}^n$ in the equation $\widehat{v}^{n+1} = \rho(\xi)\widehat{v}^n$, where $\widehat{v}^{n+1} = \rho(\xi)\widehat{v}^n$ is the discrete Fourier transform of the discrete scheme.

For simplification, we can get the discrete Fourier transform of the difference scheme by replace v_j^n in the difference scheme by

$$v_j^n = \widehat{v}^n \exp(jI\xi), I = \sqrt{-1}. \tag{12}$$

However, the difference scheme $v^{n+1} = Qv^n$ is stable with respect to the $\ell_{2,h}$ norm if and only if there exist positive constants τ_0, h_0, and $\mathtt{C}$ such that

$$|\rho(\xi)| \le 1 + \mathtt{C}\tau, \tag{13}$$

for $0 < \tau \le \tau_0, 0 < h \le h_0$, and all $\xi \in [-\pi, \pi]$, while the difference scheme that is stable under set of conditions is called conditionally stable scheme, otherwise is called unconditionally stable scheme. Here, it should be noted that when ρ satisfies inequality (13), it is said that ρ meets the von Neumann condition.

3. Modified FDDS formulation

In this section, we propose a new FDDS for solving the linear SFCE:

$$\frac{\partial u(x,t)}{\partial t} + \epsilon \frac{\partial^\beta u(x,t)}{\partial x^\beta} = 0, t \ge 0, a \le x \le b, \tag{14}$$

subject to the initial condition

$$u(x, 0) = f(x), a \le x \le b, \tag{15}$$

where ϵ is a positive parameter, β is a parameter describes the order of the space-fractional derivative under the Riemann-Liouville sense, $f(x)$ is given analytical function in term of x in $[a, b]$, and $u(x, t)$ is unknown smooth function with respect to x and t to be obtained in the domain of interest. For $\beta = 1$, this model reduces to classical linear Cauchy equation.

To achieve our goal, let $\Omega = [a, b]$ be a finite domain that is discretized with $N+1$ uniformly-spaced nodes $x_i = a + ih, i = 0, 1, \cdots, N$, where the spatial step $h = \frac{b-a}{N}$. Utilizing Riemann-Liouville fractional derivative such that

$$\frac{\partial u(x,t)}{\partial t} + \epsilon \frac{\partial}{\partial x} J_a^{1-\beta} u(x,t) = 0, 0 < \beta \le 1, \tag{16}$$

where $J_a^{1-\beta}$is the Riemann-Liouville integral with respect to x, and setting $\alpha = 1 - \beta$ so that $0 \le \alpha < 1$.

Consequently, the fractional-order α of Riemann-Liouville integral will be approximated with standard Grünwald formula based on the first derivative with central difference formula as follows:

$$J_a^\alpha u(x,t) = h^\alpha \sum_{j=0}^{N} w_j^\alpha u(x - jh, t) + o(1), \tag{17}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=x_i} = \frac{u(x_{i+1}, t) - u(x_{i-1}, t)}{2h} + \mathcal{O}(h^2). \tag{18}$$

By applying the FDDS in evaluating equation (16) at $x = x_i$ and using equations (17) and (18), it yields that

$$\frac{du(x_i,t)}{dt} = -\frac{\epsilon}{2h}\left[h^\alpha \sum_{j=0}^{i+1} w_j^\alpha u(x_{i-j+1},t) - h^\alpha \sum_{j=0}^{i-1} w_j^\alpha u(x_{i-j-1},t)\right]. \tag{19}$$

Now, define a temporal partition $t_n = n\tau, n = 0,1,\cdots,$ where τ is the time step, and use the standard backward difference to approximate the temporal derivative in equation (19) such that

$$\left.\frac{du(x_i,t)}{dt}\right|_{t=t_{n+1}} = \frac{u(x_i,t_{n+1}) - u(x_i,t_n)}{\tau} + \mathcal{O}(\tau). \tag{20}$$

Letting $u_i^n \approx u(x_i,t_n)$ denote the numerical solution, then we have

$$\frac{u_i^{n+1} - u_i^n}{\tau} = -\frac{\epsilon}{2h}\left[h^\alpha \sum_{j=0}^{i+1} w_j^\alpha u_{i-j+1}^{n+1} - h^\alpha \sum_{j=0}^{i-1} w_j^\alpha u_{i-j-1}^{n+1}\right]. \tag{21}$$

Collecting like terms, we can rewrite equation (21) as follows:

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{1}{h}\sum_{j=0}^{N} b_{ij} u_j^{n+1} = 0, i = 0,1,\cdots,N, \tag{22}$$

where

$$b_{ij} = \begin{cases} \frac{\epsilon h^{\alpha-1}\left[w_{i-j+1}^\alpha - w_{i-j-1}^\alpha\right]}{2}, j < i-1, \\ \frac{\epsilon h^{\alpha-1}\left[w_2^\alpha - w_0^\alpha\right]}{2}, j = i-1, \\ \frac{\epsilon h^{\alpha-1} w_1^\alpha}{2}, j = i, \\ \frac{\epsilon h^{\alpha-1} w_0^\alpha}{2}, j = i+1, \\ 0, j > i+1. \end{cases} \tag{23}$$

The vector of numerical solution $U^n = \left[u_0^n, u_1^n, \cdots, u_N^n\right]$ can be denoted by the following form

$$(I + \tau A)U^{n+1} = U^n, \tag{24}$$

where the matrix A has the elements $a_{ij} = b_{ij}$.

The next two theorems explain that the FDDS described in equation (22) is conditionally stable and consistent with first-order accuracy in time and second-order accuracy in space.

Theorem 3. *For* $\mathrm{i} = 0,1,\cdots,\mathrm{N}$, *the numerical scheme* $\frac{\mathrm{u}_i^{n+1} - \mathrm{u}_i^n}{\tau} = \frac{1}{\mathrm{h}}\sum_{j=0}^{N} \mathrm{b}_{ij}\mathrm{u}_j^{n+1}$ *is conditionally stable.*

Proof. Consider the homogeneous scheme

$$\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{1}{h}\sum_{j=0}^{N} b_{ij} u_j^{n+1}, i = 0,1,2,\cdots,N. \tag{25}$$

Substitute $u_i^n = \widehat{u}^n \exp(iI\xi), I = \sqrt{-1}$, into numerical scheme (22) such that

$$\begin{aligned} &\widehat{u}^{n+1}\exp(iI\xi) - u^n \exp(iI\xi) = r\textstyle\sum_{j=0}^{N} b_{ij}\widehat{u}^{n+1}\exp(jI\xi), \\ &r = \tfrac{\tau}{h}, \\ &\widehat{u}^{n+1} = \rho(\xi)\widehat{u}^n, \end{aligned} \tag{26}$$

where $\rho(\xi) = \left[1 - r\sum_{j=0}^{N} b_{ij}\exp((j-i)I\xi)\right]^{-1}$ that satisfies the von Neumann condition (13) if

$$\left|1 - r\sum_{j=0}^{N} b_{ij}\exp((j-i)I\xi)\right| \geq 1. \tag{27}$$

By using the reverse triangle inequality, we have

$$\left|1 - r\sum_{j=0}^{N} b_{ij}\exp((j-i)I\xi)\right| \geq \left|1 - \left|r\sum_{j=0}^{N} b_{ij}\exp((j-i)I\xi)\right|\right|. \tag{28}$$

Therefore, von Neumann condition will be satisfied whenever

$$\left|1 - \left|r\sum_{j=0}^{N} b_{ij}\exp((j-i)I\xi)\right|\right| \geq 1, \tag{29}$$

that is, either $1 - \left|r\sum_{j=0}^{N} b_{ij}\exp((j-i)I\xi)\right| \geq 1$, which is impossible to behold, or $1 - \left|r\sum_{j=0}^{N} b_{ij}\exp((j-i)I\xi)\right| \leq -1$, which is equivalent to

$$\left|\sum_{j=0}^{N} b_{ij}\exp((j-i)I\xi)\right| \geq \frac{2}{r}. \tag{30}$$

Hence, the symbol $\rho(\xi)$ of numerical scheme satisfies the von Neumann condition if

$$\left|\sum_{j=0}^{N} b_{ij}\exp((j-i)I\xi)\right| \geq \frac{2}{r}, \forall m = 0,1,2,\cdots,N. \tag{31}$$

which means that the numerical scheme is conditionally stable.

Theorem 4. *For* $\mathrm{i} = 0,1,\cdots,\mathrm{N}$, *the numerical scheme* $\frac{\mathrm{u}_i^{n+1} - \mathrm{u}_i^n}{\tau} = \frac{1}{\mathrm{h}}\sum_{j=0}^{N} \mathrm{b}_{ij}\mathrm{u}_j^{n+1}$ *is consistent with second-order accuracy in space and first-order in time.*

Proof. By using equations (17), (18), and (20), the fractional Cauchy equation (16) can be rewritten at $(\mathrm{x_i}, \mathrm{t_{n+1}})$ as:

$$\begin{aligned} \frac{u_i^{n+1} - u_i^n}{\tau} + \mathcal{O}(\tau) = -\frac{\epsilon}{2h}\Bigg[&h^\alpha \sum_{j=0}^{i+1} w_j^\alpha u_{i-j+1}^{n+1} + o(1) \\ &-h^\alpha \sum_{j=0}^{i-1} w_j^\alpha u_{i-j-1}^{n+1} + o(1)\Bigg] + \mathcal{O}\left(h^2\right). \end{aligned} \tag{32}$$

Consequently, we have

$$\left.\frac{\partial u(x_i,t)}{\partial t}\right|_{t=t_{n+1}} = -\epsilon\frac{\partial}{\partial x}\left[J_a^{1-\beta}u(x,t_{n+1})\right]\Big|_{x=x_i}. \tag{33}$$

4. Numerical experiments

In this section, in order to solve the SFCE (1) and (2) numerically using the FDDS, the equation is presented in a discrete specific form. Anyhow, we consider three illustrated examples to demonstrate the performance and efficiency of the proposed algorithm. The computations are performed by Wolfram-Mathematica software 11.

Example 1:. Consider the following homogeneous SFCE:

$$\frac{\partial u(x,t)}{\partial t} + \epsilon \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = 0, \tag{34}$$

subject to the initial condition

$$u(x,0) = \sin(\pi x), \tag{35}$$

where $\epsilon = 1 \times 10^{-3}$, $t \geq 0$, $x \in [1,4]$, and $0 < \beta \leq 1$. In particular, the exact solution of IVPs (34) and (35) at $\beta = 1$ is $u(x,t) = \sin(\pi(x - \epsilon t))$.

Following the FDD algorithm, using $h = 0.05$ and $\tau = 0.01$, the numerical results of FDDS with varying fractional order β such that $\beta \in \{0.75, 0.85, 0.95, 1\}$ compared with exact solution are given in Table 1 at the time $t = 0.5$ and $x \in [1, 1.5]$. In the light of showing the agreement between the FDDS and exact solutions, the absolute error of IVPs (34) and (35) are listed in Table 2 for $\beta = 1$ when $t = 0.5$ and $x \in [1, 1.5]$ with $h = 0.1$. Table 3 is devoted to the FDDS approximate solutions at $\beta = 0.95$ with varying times t such that $t = 0.5$ and $t = 1.0$ over the interval $[1, 1.5]$ with $h = 0.05$. From these tables, it can be noted that the FDDS approximate solutions are in good agreement with the exact solutions over the domain of interest. Anyhow, more iteration leads to more accurate solutions. For further analysis, the 2D-plot of the FDDS and exact solution for Example 1 are drawn in Fig. 1 at $t = 0.5$ and $x \in [1, 3.5]$. While, the surface plot of the approximate solution at $\beta = 0.95$ is shown in Fig. 2.

Example 2:. Consider the following nonhomogeneous SFCE:

$$\frac{\partial u(x,t)}{\partial t} + \epsilon \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = t, \tag{36}$$

subject to the initial condition

$$u(x,0) = \sin(\pi x), \tag{37}$$

where $\epsilon = 0.1$, $t \geq 0$, $x \in [1,4]$, and $0 < \beta \leq 1$. In particular, the exact solution of IVPs (36) and (37) at $\beta = 1$ is $u(x,t) = \frac{1}{2}(t^2 + 2\sin(\pi(x - \epsilon t)))$.

Table 2 Absolute errors for Example 1 at $\beta = 1$.

x	Exact	FDDS	Absolute Error
1.1	−0.307523	−0.309013	1.49065×10^{-3}
1.2	−0.586514	−0.587782	1.26842×10^{-3}
1.3	−0.808093	−0.809015	0.92203×10^{-3}
1.4	−0.950570	−0.951055	0.48539×10^{-3}
1.5	−0.999999	−1.000000	1.23369×10^{-6}

Table 3 FDDS of Example 1 at $\beta = 0.95$ with varying time T.

x	$T = 0.5$	$T = 1.0$
1.05	−0.156431	−0.156428
1.10	−0.309014	−0.309010
1.15	−0.454008	−0.454027
1.20	−0.587832	−0.587880
1.25	−0.707186	−0.707267
1.30	−0.809130	−0.809245
1.35	−0.891152	−0.891300
1.40	−0.951232	−0.951412
1.45	−0.987892	−0.988099
1.50	−1.000230	−1.000460

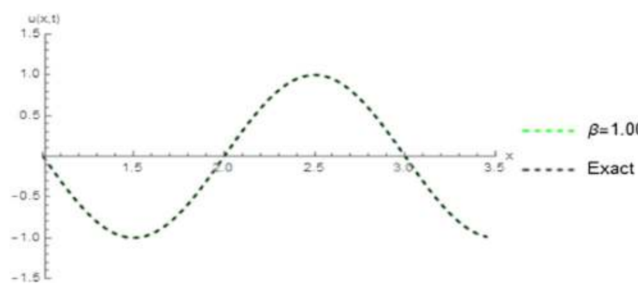

Fig. 1 FDDS and exact solutions for Example 1 at $t = 0.5$.

Following the FDDS algorithm, using $h = 0.05$ and $\tau = 0.01$, the numerical results of the FDDS and exact solutions are plot in Fig. 3 at time $t = 0.5$ and $x \in [1, 3.5]$. While Fig. 4 displays the approximate solutions of IVPs (36) and

Table 1 Numerical results for Example 1 at $t = 0.5$ with varying β.

x	Exact	$\beta = 1$	$\beta = 0.95$	$\beta = 0.85$	$\beta = 0.75$
1.05	−0.154883	−0.156431	−0.156431	−0.156432	−0.156432
1.10	−0.307523	−0.309013	−0.309014	−0.309014	−0.309015
1.15	−0.452590	−0.453987	−0.454008	−0.454031	−0.454036
1.20	−0.586514	−0.587782	−0.587832	−0.587886	−0.587900
1.25	−0.705995	−0.707104	−0.707186	−0.707276	−0.707301
1.30	−0.808093	−0.809015	−0.809130	−0.809257	−0.809294
1.35	−0.890292	−0.891005	−0.891152	−0.891317	−0.891366
1.40	−0.950570	−0.951055	−0.951232	−0.951433	−0.951495
1.45	−0.987441	−0.987688	−0.987892	−0.988124	−0.988199
1.50	−0.999999	−1.000000	−1.000230	−1.000490	−1.000570

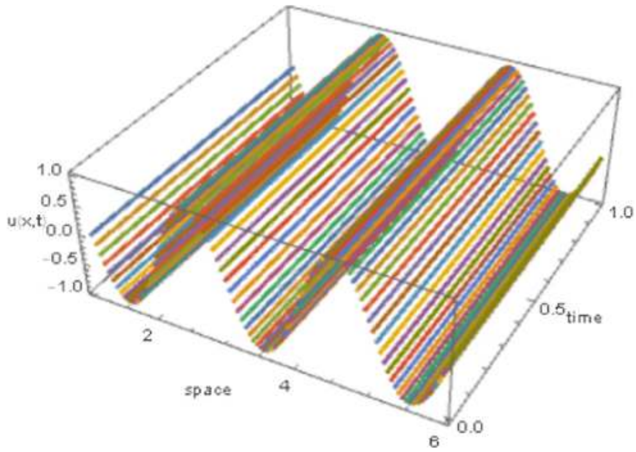

Fig. 2 Surface plot of FDDS solution for Example 1 at $\beta = 0.95$.

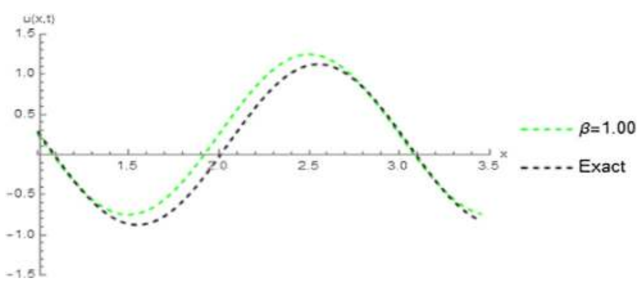

Fig. 3 FDDS and exact solutions for Example 2 at $t = 0.5$.

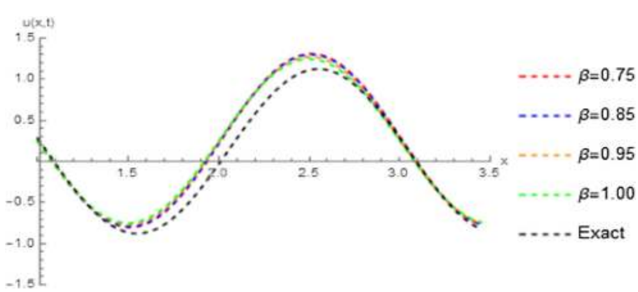

Fig. 4 Solution behavior of approximate solutions for Example 2 for different values of fractional order β.

(37) for different values of fractional-order β such that $\beta \in \{0.75, 0.85, 0.95, 1\}$ at time $t = 0.5$ and $x \in [1, 3.5]$. From these graphs, it can be concluded that the behavior of the FDDS approximate solutions are in good agreement with each other for different values of fractional order, which are very closed at the initial and endpoints. To show the effect of the time-level of the proposed scheme, the 2-dimintional plot of the FDDS solutions with varying time T such that $T = 0.5$ and $T = 1.0$ are shown in Fig. 5. Further, Fig. 6 depicts the surface plot of the approximate solution at $\beta = 0.95$.

Example 3:. Consider the following SFCE:

$$\frac{\partial u(x,t)}{\partial t} + \epsilon \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = 0, \tag{38}$$

subject to the initial condition

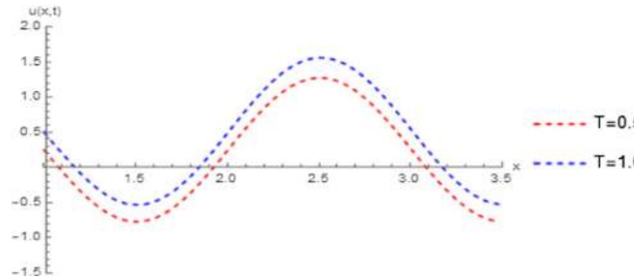

Fig. 5 Solution behavior of approximate solutions of Example 2 with different time-level T.

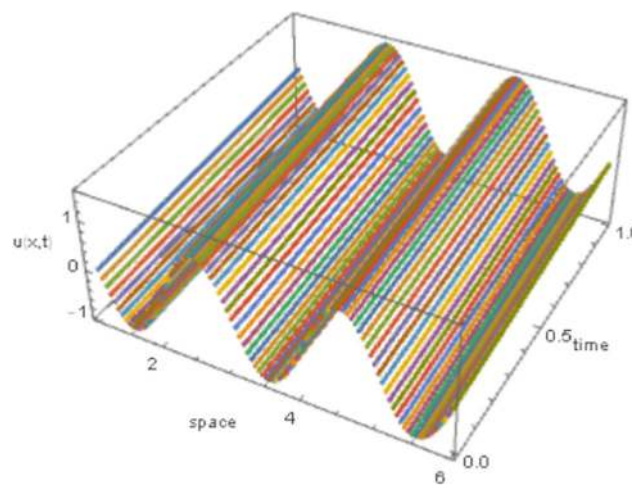

Fig. 6 Surface plot of FDDS solution for Example 2 at $\beta = 0.95$.

$$u(x,0) = e^{\xi x}, \tag{39}$$

where $\epsilon = 0.1$, $\xi = 1.1771243444677$, $t \geq 0$, $x \in [-2, 1]$, and $0 < \beta \leq 1$. In particular, the exact solution of IVPs (38) and (39) at $\beta = 1$ is given by $u(x,t) = e^{\xi(x - \epsilon t)}$.

Following the FDDS algorithm, using $h = 0.0625$ and $\tau = 0.01$, the numerical results of the exact and FDDS for different values of fractional order β such that $\beta \in \{0.75, 0.85, 0.95, 1\}$ are given in Table 4 at the time $t = 0.5$ and $x \in [-2, -1.25]$. Table 5 is devoted to the FDDS approximate solutions at $\beta = 0.95$ with varying times t such that $t = 0.5$ and $t = 1$ over the interval $[-2, -1.5]$ with $h = 0.0625$. From the tables, it can be noted that the FDDS approximate solutions are in good agreement with the exact solutions over the domain of interest. Anyhow, more iteration leads to more accurate solutions. For further analysis, the 2D-plot of the FDDS and exact solution of IVPs (38) and (39) are drawn in Fig. 7 at $t = 0.5$ and $x \in [-2, 1]$. Fig. 8 displays the approximate solutions of IVPs (38) and (39) for different values of fractional-order β such that $\beta \in \{0.75, 0.85, 0.95, 1\}$ at time $t = 0.5$ and $x \in [-2, 1]$. While, the surface plot of the FDDS approximate solution at $\beta = 0.95$ is shown in Fig. 9. From these graphs, it can be concluded that the behavior of the FDDS approximate solutions are in good agreement with each other at different values of β. (See Fig. 10.)

Table 4 Numerical results for Example 3 at $t = 0.5$ with varying β.

x	Exact	$\beta = 1$	$\beta = 0.95$	$\beta = 0.85$	$\beta = 0.75$
−2.0000	0.089537	0.094808	0.094822	0.094847	0.094868
−1.9375	0.096373	0.102192	0.102184	0.102175	0.102171
−1.8750	0.103730	0.109994	0.111023	0.112139	0.112442
−1.8125	0.111649	0.118391	0.119907	0.121624	0.122163
−1.7500	0.120173	0.127429	0.129290	0.131461	0.132197
−1.6875	0.129347	0.137158	0.139306	0.141862	0.142774
−1.6250	0.139222	0.147629	0.150043	0.152954	0.154028
−1.5625	0.149851	0.158900	0.161572	0.164827	0.166060
−1.5000	0.161291	0.171031	0.173965	0.177565	0.178954
−1.4375	0.173605	0.184088	0.187291	0.191244	0.192792
−1.3750	0.186859	0.198142	0.201627	0.205946	0.207658
−1.3125	0.201124	0.213269	0.217050	0.221754	0.223637
−1.2500	0.216479	0.229551	0.233647	0.238756	0.240817

Table 5 FDDS of Example 3 at $\beta = 0.95$ with varying time t.

x	$t = 0.5$	$t = 1$
−2.0000	0.094822	0.094677
−1.9375	0.102184	0.102151
−1.8750	0.111023	0.112044
−1.8125	0.119907	0.121425
−1.7500	0.12929	0.131169
−1.6875	0.139306	0.141487
−1.6250	0.150043	0.152500
−1.5625	0.161572	0.164299
−1.5000	0.173965	0.176962

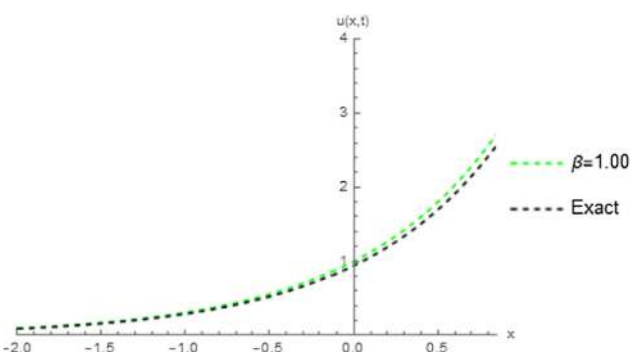

Fig. 7 FDDS and exact solutions for Example 3 at $t = 0.5$.

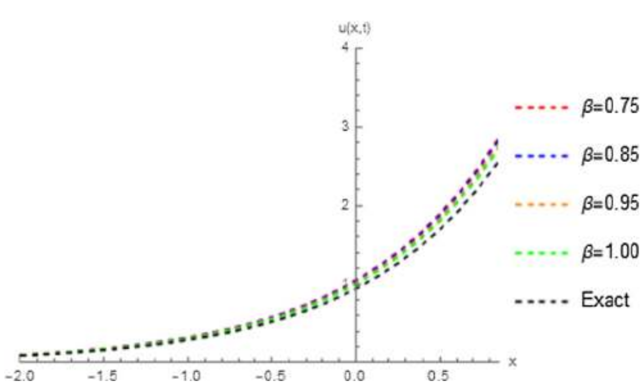

Fig. 8 Solution behavior of approximate solutions for Example 3 for different values of fractional order β.

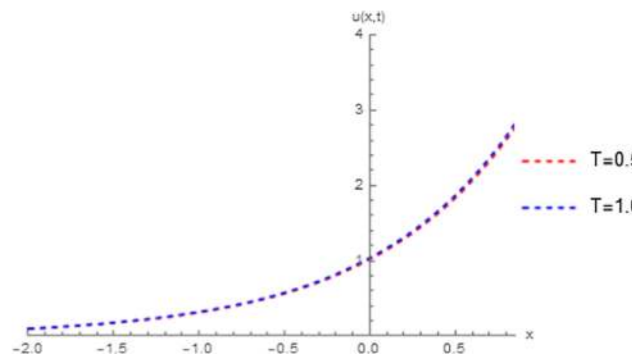

Fig. 9 Solution behavior of approximate solutions of Example 3 with different time-level T.

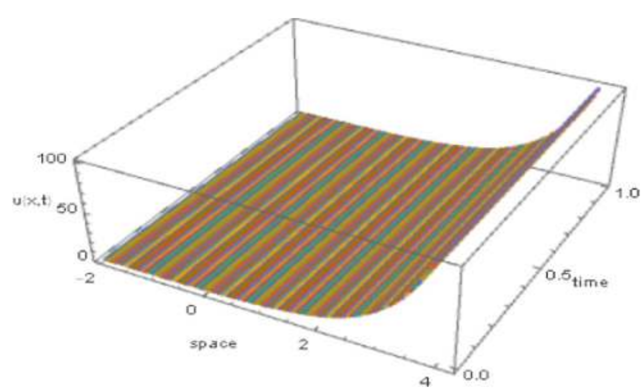

Fig. 10 Surface plot of FDDS solution for Example 3 at $\beta = 0.95$.

5. Conclusion

In this work, a new FDDS with supporting analysis has been developed for solving the SFCE with and without source term under the Grünwald-Letnikov fractional formula. The approximate solution has been obtained based on the implicit discretization scheme. Stability and consistency were discussed, which turns out that the proposed scheme is conditionally stable, first-order accurate in time, and second-order accurate in space. Three numerical applications have been added to demonstrate the ability and efficiency of the method. The results indicated that the approximate solutions are coinciding with each other at the selected nods and parameters. From the obtained results, we can conclude that the modified FDDS is a systematic and suitable scheme to address numerous fractional problems across physics, applied sciences, and engineering. The calculations have been performed by using the Wolfram-Mathematica 11.

Declaration of Competing Interest

The authors declare that they have no conflicts of interest.

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