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Smooth expansion to solve high-order linear conformable fractional PDEs via residual power series method: Applications to physical and engineering equations

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ABSTRACT

We present a fractional series solution (FSS) for a class of higher-order linear fractional PDEs. The fractional derivative in this class is considered in the conformable fractional derivative (Co-FD) sense. An appropriate expansion was introduced to reach an FSS that is consistent with the target equations in this research. The residual power series technique is used to determine the coefficients of the FSS. Five applications are tested to verify the effectiveness of the used method, as well as to compare the current results with the previous results for the same applications in which the fractional derivative was considered in the Caputo sense. Numerical and graphical comparisons are made to determine the compatibility of the behavior of the solution in the case of the use of the concept of Co-FD as a suitable alternative to the use of the concept of Caputo fractional derivative (Ca-FD) in the modeling of natural phenomena.

1. Introduction

Fractional PDEs have been employed widely to describe several phenomena in physics and aid in their solutions such as, fluid flow, electrodynamics, quantum mechanics, rheology, damping laws and diffusion processes [1-9]. It is a quite difficult to obtain the exact solutions of problems in real-life using PDEs of fractional order and complicated mathematical techniques are usually required. Therefore, approximated methods (analytical or numerical) of such problems are used such as reproducing kernel method, Adomian decomposition method (ADM), homotopy perturbation method (HPM), homotopy analysis method (HAM), fractional transform method (FTM), deep collocation method, artificial neural network methods and other methods [9-28]. Recently, fractional series solutions (FSSs) have been widely used to solve many classes of these problems [29-35]. Unfortunately, there is no uniform form to the FSS for all differential equations. Each equation has a different FSS form than the other. The generic term of the FSS may be of the form $f_n(x;\alpha)t^{n\alpha}$ or the form $f_n(x;\alpha)t^{n\alpha+m}$, in case the solution of the PDE is a multivariable function, where, α is the order of the fractional derivative, x and t are the independent variables of the PDE, f_n is the coefficients of series, $m = 0, 1, 2, \cdots, k \in \mathbb{N}$ and $n = 0, 1, 2, \cdots$. In fact, the FSS form depends on several factors, including the fractional derivative type, the initial conditions form, and the form of the coefficients in the PDE.

There are many types or definitions of fractional derivatives such as the definition of Grünwald–Letnikov, the definition of Riemann-Liouville, the definition of Caputo and other definitions [7-9]. In fact, fractional derivatives provide an excellent tool for

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the description of memory and hereditary properties of various materials, processes and to find more accurate results in modeling systems with complex structure and systems based on diffusion processes. Recently, a new definition of fractional derivative was developed by R. Khalil et al. [36] which is called the "conformable fractional derivative (Co-FD)" and it is very easily computed compared with the previous definitions. The Co-FD has many advantages over other definitions of fractional derivatives, most notably, it simulates most properties and concepts of the ordinary derivative such as: product, quotient and chain rules while the other fractional definitions fail to satisfy these rules. Therefore, it has received a lot of attention from many researchers and many applications have been remodeled using this definition, such as, Laguerre and Lane-Emden [37], quantum mechanics [38], heat equation [39], Bernoulli and Riccati equations [40], Wu–Zhang system [41], KdV-Burgers Equation [42], and time-fractional nonlinear dispersive PDEs [43], Schrödinger equations [44].

In this research paper, a new expansion is developed, and a new method, called residual power series (RPS), is used to construct an FSS for a class of higher-order linear conformable fractional PDEs (LCF-PDEs) given by:

$$T^{\alpha}_{\omega}\Theta(\tau,\omega) = \phi(\tau,\omega) + \sum_{j=0}^{m-1}\left(h_j(\tau)(\omega-\omega_0)^j T^j_{\omega}\left[L^j_{\tau}\Theta(\tau,\omega)\right]\right), \quad \tau\in I, \omega \ge \omega_0, \tag{1.1}$$

subject to the nonhomogeneous ordinary initial conditions

$$T^j_{\omega}\Theta(\tau,\omega_0) = \varphi_j(\tau), \tau\in I, j = 0,1,2,\cdots,m-1, \tag{1.2}$$

where $T^{\alpha}_{\omega}, \omega \ge \omega_0$ is a conformable fractional differential operator of order $\alpha\in(m-1,m], m\in\mathbb{N}$ (the set of integer numbers), L^j_{τ} is a linear differential operator with respect to the independent variable τ, $\varphi_j(\tau), h_j(\tau)$ are given analytic functions on I and $\phi(\tau,\omega)$ is a multivariable function that can be expanded as we will see later.

The RPS technique [29-32,42-44] is easy and effective to find a FSS for nonlinear and linear equations without, perturbation, linearization and discretization by a sequence of algebraic equations. Moreover, the RPS can be applied to compute analytical solutions of fractional PDEs. The series solution is obtained by using the concept of residual error, where this solution and their fractional derivatives are valid in the given domain for each multidimensional variable and each arbitrary point. The RPS method has been applied to solve many types of differential equations such as fractional KdV-Burgers equation in Caput and Conformable sense [29,42], nonlinear time-fractional dispersive PDEs [30,43], Multi-Energy Groups of Neutron Diffusion Equations [31], fractional multi-pantograph system [32], nonlinear time-fractional Schrödinger equations [44].

The solution of the higher-order LF-PDEs are constantly require due to physical and engineering interests, where it is very often that the analytical solution of such equations cannot be obtained. However, we abled to provide analytical and numerical solutions for the higher-order LCF-PDEs by using the RPS method. In some cases, we obtained the exact solution for the higher-order LCF-PDEs that mentioned in (1.1) and (1.2) as we will see in Section 4.

The research given in this paper is outline in five sections. Some important definitions and theorems from the theory of conformable fractional calculus are given and a fractional power series in the Co-FD sense is presented in Section 2. The main concept of the RPS method to build a FSS is presented in Section 3. In Section 4, the proposed method is validated through five applications. Finally, the conclusion is presented.

2. On conformable fractional calculus

The definition of Co-FD in this paper is adopted for the concept of fractional derivative, a simple new definition that contains most of the properties of the classical derivative. The following are some of the results and definitions required from the theory of conformable fractional calculus and can be found in more detail in [36-44].

Definition 2.1 (*[36,43]*). An ω- Co-FD of order $\alpha\in(m-1,m], m\in\mathbb{N}$ of a function $\Theta(\tau,\omega): I\times[s,\infty)\to\mathbb{R}$ is defined by

$$T^{\alpha}_{\omega}\Theta(\tau,\omega) = \partial^{\alpha}_{\omega}\Theta(\tau,\omega) = \lim_{\varepsilon\to 0}\frac{\partial^{m-1}_{\omega}\Theta\left(\tau,\omega+\varepsilon(\omega-s)^{m-\alpha}\right)-\partial^{m-1}_{\omega}\Theta(\tau,\omega)}{\varepsilon}, \omega > s,$$
$$T^{\alpha}_{\omega}\Theta(\tau,s) = \lim_{\omega\to s^+} T^{\alpha}_{\omega}\Theta(\tau,\omega), \tag{2.1}$$

provided the limits exist and $\partial^k\Theta/\partial\omega^k,\ k=1,2,\cdots,m-1$ is defined for $\omega > s$.

By the previous definition, one can easily prove the properties in the following Lemma.

Lemma 2.1 (*[36,43]*). *Let $\alpha\in(m-1,m], m\in\mathbb{N}, \lambda$ is a constant. Then*

(1) $T^{\alpha}_{\omega}(\lambda) = 0$,
(2) $T^{\alpha}_{\omega}(\Theta(\tau,\omega)+\varphi(\tau,\omega)) = T^{\alpha}_{\omega}\Theta(\tau,\omega)+T^{\alpha}_{\omega}\varphi(\tau,\omega)$,
(3) $T^{\alpha}_{\omega}(\lambda\Theta(\tau,\omega)) = \lambda T^{\alpha}_{\omega}\Theta(\tau,\omega)$,
(4) $T^{\alpha}_{\omega}(\tau,\omega) = (\omega-s)^{m-\alpha}\partial^{m}_{\omega}\Theta(\tau,\omega)$,
(5) $T^{\alpha}_{\omega}(\omega-s)^{\gamma} = \begin{cases}\prod_{k=0}^{m-1}(\gamma-k)(\omega-s)^{\gamma-\alpha}, \gamma\notin\{0,1,2,\cdots,m-1\}\\ 0, \gamma\in\{0,1,2,\cdots,m-1\}\end{cases}$.

Remark 2.1. If $\gamma\notin\{\cdots,-3,-2,-1,0,1,2,\cdots,m-1\}$, then $T^{\alpha}_{\omega}(\omega-s)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-m)}(\omega-s)^{\gamma-\alpha}$, $\alpha\in(m-1,m]$.

Definition 2.2 (*[36,43]*). An ω-conformable fractional integral (CFI) of order $\alpha\in(m-1,m], m\in\mathbb{N}$ of a function $\Theta(\tau,\omega): I\times[s,\infty)\to\mathbb{R}$ is defined by

$$I^{\alpha}_{\omega}\Theta(\tau,\omega) = \frac{1}{(m-1)!}\int_s^{\omega}\frac{(\omega-x)^{m-1}\Theta(\tau,x)}{(x-s)^{m-\alpha}}dx, \alpha>0, \omega>s,$$
$$I^{0}_{\omega}\Theta(\tau,\omega) = \Theta(\tau,\omega). \tag{2.2}$$

Lemma 2.2 (*[36,43]*). *If $\alpha\in(m-1,m], m\in\mathbb{N}$, λ is a constant and $\alpha+\gamma-m>-1$, then*

(1) $I^{\alpha}_{\omega}(\omega-s)^{\gamma} = \frac{\Gamma(1+\gamma+\alpha-m)}{\Gamma(1+\gamma+\alpha)}(\omega-s)^{\gamma+\alpha}$,
(2) $I^{\alpha}_{\omega}(\lambda) = \lambda\frac{\Gamma(1+\alpha-m)}{\Gamma(1+\alpha)}(\omega-s)^{\alpha}$,

Lemma 2.3 (*[36,43]*). *If $\alpha\in(m-1,m], m\in\mathbb{N}$, and $\Theta(\tau,\omega): I\times[s,\infty)\to\mathbb{R}$ such that $T^k_{\omega}\Theta = \partial^k\Theta/\partial\omega^k$, $k=1,2,\cdots,m-1$ is defined for $\omega>s$., then*

(1) $T^{\alpha}_{\omega}I^{\alpha}_{\omega}\Theta(\tau,\omega) = \Theta(\tau,\omega)$,
(2) $I^{\alpha}_{\omega}T^{\alpha}_{\omega}\Theta(\tau,\omega) =$
$I^{m}_{\omega}T^{m}_{\omega}\Theta(\tau,\omega) = \Theta(\tau,\omega) - \sum_{k=0}^{m-1}\left(T^k_{\omega}\Theta\right)(\tau,\omega)\frac{(\omega-s)^k}{k!}, \omega>s.$

To be able to obtain the series solution analytically for Eqs. (1.1) and (1.2), some definitions and theorems are given below related to the power series of fractional order in the sense of Co-FD definition.

Definition 2.3. A power series (PS) of the form

$$\sum_{r=0}^{\infty}\sum_{k=0}^{m-1} f_{rk}(\tau)(\omega-s)^{k+r\alpha}, \omega \geq s, 0 \leq m-1 < \alpha \leq m, \tag{2.3}$$

is called a multivariable fractional PS at $\omega = s$, where ω is a variable and $f_{rk}(\tau)$ are functions of τ called the coefficients of the PS.

As the classical PS, all terms of Eq. (2.3), except the first term, vanish when$\omega = s$. Therefore, the multivariable fractional PS in Eq. (2.3) is convergent when $\omega = s$. Moreover, for $\omega > s$, we can prove that the series in Eq. (2.3) may be convergent on an infinite interval of the form (s,∞) or on a finite interval (s,R) with radius of convergence, ∞ or R, respectively.

Theorem 2.1 (*Conformable fractional multivariable Taylor's series*). *Let $\alpha \in (m-1, m]$ and assume that $\Theta(\tau,\omega)$ has a multivariable fractional PS at s as follows:*

$$\Theta(\tau,\omega) = \sum_{r=0}^{\infty}\sum_{k=0}^{m-1} f_{rk}(\tau)(\omega-s)^{k+r\alpha}, s \leq \omega < s+R. \tag{2.4}$$

Let $\Theta(\tau,\omega)$ be continuous on $I \times [s, s+R)$, $T_\omega^{r\alpha}\Theta(\tau,\omega)$ is continuous on $I \times (s, s+R)$, and $T_\omega^k T_\omega^{r\alpha}\Theta(\tau,\omega)$ is defined on $I \times (s, s+R)$ for $k = 1,2,\cdots,m-1, r = 0,1,2,\cdots$, where $T_\omega^{r\alpha}= T_\omega^\alpha \cdot T_\omega^\alpha \cdots T_\omega^\alpha$ (r-times).Then then coefficients, $f_{rk}(\tau)$, of Eq. (2.4) are given by the formula

$$f_{rk} = \frac{1}{k!}\begin{cases} T_\omega^k\Theta(\tau,s), r=0 \\ \prod_{n=1}^{r}\frac{\Gamma(k+1+n\alpha-m)}{\Gamma(k+1+n\alpha)} T_\omega^k T_\omega^{r\alpha}\Theta(\tau,s), r=1,2,3,\cdots \end{cases}, k=0,1,2,\cdots,m-1 \tag{2.5}$$

Proof. Assume that $\Theta(\tau,\omega)$ is any multivariable function that can be represented by the multivariable fractional PS in Eq. (2.3). First of all, note that if we put $\omega = s$ in Eq. (2.4), all terms after the first one vanishes, and we get

$$f_{00} = \Theta(\tau,s). \tag{2.6}$$

If we apply the operator T_ω^1 on the Eq. (2.4), then we have

$$\begin{aligned} T_\omega^1\Theta(\tau,\omega) &= f_{01}(\tau) + 2f_{02}(\tau)(\omega-s) + \cdots \\ &+ (m-1)f_{0(m-1)}(\tau)(\omega-s)^{m-2} \\ &+ \alpha f_{10}(\tau)(\omega-s)^{\alpha-1} + (1+\alpha)f_{11}(\tau)(\omega-s)^{\alpha} \\ &+ \cdots + (m-1+\alpha)f_{1(m-1)}(\tau)(\omega-s)^{m-2+\alpha} \\ &+ \cdots, \end{aligned} \tag{2.7}$$

and substitute $\omega = s$ into Eq. (2.7), we obtain

$$f_{01} = \left(T_\omega^1\Theta\right)(\tau,s). \tag{2.8}$$

Again, if we apply the operator T_ω^1 on Eq. (2.7), then we get

$$\begin{aligned} T_\omega^2\Theta(\tau,\omega) &= 2f_{02}(\tau) + 6f_{03}(\tau)(\omega-s) + \cdots \\ &+ (m-1)(m-2)f_{0(m-1)}(\tau)(\omega-s)^{m-3} \\ &+ \alpha(\alpha-1)f_{10}(\tau)(\omega-s)^{\alpha-2} \\ &+ (1+\alpha)\alpha f_{11}(\tau)(\omega-s)^{\alpha-1} + \cdots \\ &+ (m-1+\alpha)(m-2+\alpha)f_{1(m-1)}(\tau)(\omega-s)^{m-3+\alpha} \\ &+ \cdots. \end{aligned} \tag{2.9}$$

Substituting $\omega = s$ into Eq. (2.9) gives

$$f_{02}(\tau) = \frac{1}{2}\left(T_\omega^2\Theta\right)(\tau,s). \tag{2.10}$$

The pattern is clear, if we operate T_ω^1 k-times on the series in Eq. (2.4) and substitute $\omega = s$ in the resulting equation, then we obtain

$$f_{0k}(\tau) = \frac{1}{k!}\left(T_\omega^k\Theta\right)(\tau,s). \tag{2.11}$$

Back to Eq. (2.4), apply the operator T_ω^α on both sides and use part (5) of Lemma 2.1 and Remark 2.1, then we have

$$\begin{aligned} T_\omega^\alpha\Theta(\tau,\omega) &= \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m)}f_{10}(\tau) \\ &+ \frac{\Gamma(2+\alpha)}{\Gamma(2+\alpha-m)}f_{11}(\tau)(\omega-s) + \cdots \\ &+ \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)}f_{1(m-1)}(\tau)(\omega-s)^{m-1} \\ &+ \frac{\Gamma(1+2\alpha)}{\Gamma(1+2\alpha-m)}f_{20}(\tau)(\omega-s)^{\alpha} \\ &+ \frac{\Gamma(2+2\alpha)}{\Gamma(2+2\alpha-m)}f_{21}(\tau)(\omega-s)^{1+\alpha} + \cdots \\ &+ \frac{\Gamma(m+2\alpha)}{\Gamma(2\alpha)}f_{2(m-1)}(\tau)(\omega-s)^{m-1+\alpha} + \cdots. \end{aligned} \tag{2.12}$$

Note that the first m terms of Eq. (2.4) are vanishing depending on the part (5) of Lemma 2.1.

Now, if we substitute $\omega = s$ into Eq. (2.12), then we obtain on

$$f_{10}(\tau) = \frac{\Gamma(1+\alpha-m)}{\Gamma(1+\alpha)}\left(T_\omega^\alpha\Theta\right)(\tau,s). \tag{2.13}$$

On the other hand, if we apply the operator T_ω^1 on Eq. (2.12), then we have

$$\begin{aligned} T_\omega^1 T_\omega^\alpha\Theta(\tau,\omega) &= \frac{\Gamma(2+\alpha)}{\Gamma(2+\alpha-m)}f_{11}(\tau) + 2 \\ &\times \frac{\Gamma(3+\alpha)}{\Gamma(3+\alpha-m)}f_{12}(\tau)(\omega-s) + \cdots \\ &+ (m-1)\frac{\Gamma(m+\alpha)}{\Gamma(\alpha)}f_{1(m-1)}(\tau)(\omega-s)^{m-2} \\ &+ \alpha\frac{\Gamma(1+2\alpha)}{\Gamma(1+2\alpha-m)}f_{20}(\tau)(\omega-s)^{\alpha-1} \\ &+ (1+\alpha)\frac{\Gamma(2+2\alpha)}{\Gamma(2+2\alpha-m)}f_{21}(\tau)(\omega-s)^{\alpha} \\ &+ \cdots \\ &+ (m-1+\alpha)\frac{\Gamma(m+2\alpha)}{\Gamma(2\alpha)}f_{2(m-1)}(\tau)(\omega-s)^{m-2+\alpha} \\ &+ \cdots. \end{aligned} \tag{2.14}$$

Substituting $\omega = s$ into Eq. (2.14) gives

$$f_{11}(\tau) = \frac{\Gamma(2+\alpha-m)}{\Gamma(2+\alpha)}\left(T_\omega^1 T_\omega^\alpha\Theta\right)(\tau,s). \tag{2.15}$$

Applying the operator T_ω^1 on the series in Eq. (2.14) gives us

$$T_\omega^2 T_\omega^\alpha \Theta(\tau,\omega) = 2\frac{\Gamma(3+\alpha)}{\Gamma(3+\alpha-m)} f_{12}(\tau) + 6\frac{\Gamma(4+\alpha)}{\Gamma(4+\alpha-m)} f_{13}(\tau)(\omega-s) + \cdots + (m-1)(m-2)\frac{\Gamma(m+\alpha)}{\Gamma(\alpha)} f_{1(m-1)}(\tau)(\omega-s)^{m-3} + \alpha(\alpha-1)\frac{\Gamma(1+2\alpha)}{\Gamma(1+2\alpha-m)} f_{20}(\tau)(\omega-s)^{\alpha-2} + (1+\alpha)\alpha\frac{\Gamma(2+2\alpha)}{\Gamma(2+2\alpha-m)} f_{21}(\tau)(\omega-s)^{\alpha-1} + \cdots + (m-1+\alpha)(m-2+\alpha)\frac{\Gamma(m+2\alpha)}{\Gamma(2\alpha)} f_{2(m-1)}(\tau) (\omega-s)^{m-3+\alpha} + \cdots. \quad (2.16)$$

Again, if we put$\omega = s$ into Eq. (2.16), then the result is

$$f_{12}(\tau) = \frac{\Gamma(3+\alpha-m)}{2\Gamma(3+\alpha)} \left(T_\omega^2 T_\omega^\alpha \Theta\right)(\tau, s). \quad (2.17)$$

Similar to the previous pattern, if we continue to operate T_ω^1 k-times and substitute $\omega = s$, we can obtain

$$f_{1k}(\tau) = \frac{\Gamma(k+1+\alpha-m)}{k!\Gamma(k+1+\alpha)} \left(T_\omega^k T_\omega^\alpha \Theta\right)(\tau, s), k = 0, 1, 2, \cdots, m-1. \quad (2.18)$$

Apply the operator T_ω^α on the series representation in Eq. (2.12), one can get on

$$T_\omega^{2\alpha}\Theta(\tau,\omega) = \frac{\Gamma(1+2\alpha)}{\Gamma(1+2\alpha-m)} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m)} f_{20}(\tau) + \frac{\Gamma(2+2\alpha)}{\Gamma(2+2\alpha-m)} \frac{\Gamma(2+\alpha)}{\Gamma(2+\alpha-m)} f_{21}(\tau)(\omega-s) + \cdots + \frac{\Gamma(m+2\alpha)}{\Gamma(2\alpha)} \times \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)} f_{2(m-1)}(\tau)(\omega-s)^{m-1} + \cdots, s \le \omega < s+R, \quad (2.19)$$

The substitution of $\omega = s$ into Eq. (2.19) gives

$$f_{20}(\tau) = \frac{\Gamma(1+\alpha-m)}{\Gamma(1+\alpha)} \frac{\Gamma(1+2\alpha-m)}{\Gamma(1+2\alpha)} \left(T_\omega^{2\alpha}\Theta\right)(\tau, s). \quad (2.20)$$

Again, applying the operator T_ω^1 on the series in Eq. (2.19) and substituting $\omega = s$ in the resulting formula, we will obtain

$$f_{21}(\tau) = \frac{\Gamma(2+\alpha-m)}{\Gamma(2+\alpha)} \frac{\Gamma(2+2\alpha-m)}{\Gamma(2+2\alpha)} \left(T_\omega^1 T_\omega^{2\alpha}\Theta\right)(\tau, s). \quad (2.21)$$

If we operate T_ω^k on the series in Eq. (2.19) and substitute $\omega = s$, we can have the coefficients $f_{2k}(\tau), k = 0, 1, 2, \cdots, m-1$ of Eq. (2.4) that are given in following formula:

$$f_{2k}(\tau) = \frac{\Gamma(k+1+\alpha-m)}{k!\Gamma(k+1+\alpha)} \times \frac{\Gamma(k+1+2\alpha-m)}{\Gamma(k+1+2\alpha)} \left(T_\omega^k T_\omega^{2\alpha}\Theta\right)(\tau, s). \quad (2.22)$$

Now we can see the pattern completely. However, if we operate $T_\omega^{n\alpha}$ on Eq. (2.4) and then operate T_ω^k on the resulting equation and then substitute $\omega = s$, then we get on the general form of the coefficients, $f_{rk}(\tau)$, of the series in Eq. (2.4) that given in Eq. (2.5).

Substituting this formula of $f_{rk}(\tau)$ for back into the series, we see that if $\Theta(\tau,\omega)$ has a multivariable fractional PS expansion at s, then it must be of the following form:

$$\Theta(\tau,\omega) = \sum_{r=0}^{\infty}\sum_{k=0}^{m-1} \frac{\left(T_\omega^k T_\omega^{r\alpha}\Theta\right)(\tau, s)}{\chi(r,k)k!} (\omega-s)^{k+r\alpha}, s \le \omega < s+R, \quad (2.23)$$

where

$$\chi(r,k) = \begin{cases} 1, r = 0 \\ \prod_{j=1}^{r} \frac{\Gamma(k+1+j\alpha)}{\Gamma(k+1+j\alpha-m)}, r = 1, 2, \cdots \end{cases}, \quad (2.24)$$

which is a new form of the Taylor's series formula. Moreover, in case of $m = 1$, the conformable fractional multivariable Taylor's series (CFMTS) formula (2.23) reduces to the following simple form:

$$\Theta(\tau,\omega) = \sum_{r=0}^{\infty} \frac{\left(T_\omega^{r\alpha}\Theta\right)(\tau, s)}{\alpha^r r!} (\omega-s)^{r\alpha}, s \le \omega < s+R. \quad (2.25)$$

Moreover, if $\alpha = 1$, then we obtain the following classical Taylor's series formula:

$$\Theta(\tau,\omega) = \sum_{\infty}^{r=0} \frac{\partial^r\Theta(\tau,s)}{\partial\omega^r} \frac{(\omega-s)^r}{r!}, s \leqslant \omega < s+R. \blacksquare \quad (2.26)$$

Theorem 2.2. *Let $\alpha \in (m-1, m]$ and suppose that$\Theta(\tau,\omega) \in C(I \times [s, s+R))$, $T_\omega^{r\alpha}\Theta(\tau,\omega) \in C(I \times (s, s+R))$, and $T_\omega^{r\alpha}\Theta(x,\omega)$ can be differentiated $(m-1)$-times with respect to ω on $(s, s+R)$ for $r = 0, 1, 2, \cdots, n+1$. Then*

$$I_\omega^{(n+1)\alpha} T_\omega^{(n+1)\alpha}\Theta(\tau,\omega) = \Theta(\tau,\omega) - \sum_{r=0}^{n}\sum_{k=0}^{m-1} \times \frac{\left(T_\omega^k T_\omega^{r\alpha}\Theta\right)(\tau, s)}{\chi(r,k)k!} (\omega-s)^{k+r\alpha}. \quad (2.27)$$

Proof. From the certain properties of the operator I_t^α and part (2) of Lemma 2.3, we have

$$I_\omega^{(n+1)\alpha} T_\omega^{(n+1)\alpha}\Theta(\tau,\omega) = I_\omega^{n\alpha}\left(\left(I_\omega^\alpha D_\omega^\alpha\right) T_\omega^{n\alpha}\Theta(\tau,\omega)\right) = I_\omega^{n\alpha}\left(\left(I_\omega^m T_\omega^m\right) T_\omega^{n\alpha}\Theta(\tau,\omega)\right)$$

$$= I_\omega^{n\alpha}\left(T_\omega^{n\alpha}\Theta(\tau,\omega) - \sum_{k=0}^{m-1} \frac{\left(T_\omega^k T_\omega^{n\alpha}\Theta\right)(\tau,s)}{k!} (\omega-s)^k\right)$$

$$= I_\omega^{n\alpha} T_\omega^{n\alpha}\Theta(\tau,\omega) - I_\omega^{n\alpha}\left(\sum_{k=0}^{m-1} \frac{\left(T_\omega^k T_\omega^{n\alpha}\Theta\right)(\tau,s)}{k!} (\omega-s)^k\right)$$

$$= I_\omega^{(n-1)\alpha}\left(\left(I_\omega^m T_\omega^m\right) T_\omega^{(n-1)\alpha}\Theta(\tau,\omega)\right) - \sum_{k=0}^{m-1}\left(\prod_{j=1}^{n} \frac{\Gamma(1+k+j\alpha-m)}{\Gamma(1+k+j\alpha)}\right) \frac{\left(T_\omega^k T_\omega^{n\alpha}\Theta\right)(\tau,s)}{k!} (\omega-s)^{k+n\alpha}$$

$$= I_\omega^{(n-1)\alpha}\left(T_\omega^{(n-1)\alpha}\Theta(\tau,\omega) - \sum_{k=0}^{m-1} \frac{\left(T_\omega^k T_\omega^{(n-1)\alpha}\Theta\right)(\tau,s)}{k!} (\omega-s)^k\right)$$

$$- \sum_{k=0}^{m-1} \frac{\left(T_\omega^k T_\omega^{n\alpha}\Theta\right)(\tau,s)}{\chi(n,k)k!} (\omega-s)^{k+n\alpha}$$

$$= I_{\omega}^{(n-1)\alpha} T_{\omega}^{(n-1)\alpha} \Theta(\tau,\omega) - I_{\omega}^{(n-1)\alpha} \left(\sum_{k=0}^{m-1} \frac{\left(T_{\omega}^{k} T_{\omega}^{(n-1)\alpha} \Theta\right)(\tau,s)}{k!} (\omega - s)^{k} \right) - \sum_{k=0}^{m-1} \frac{\left(T_{\omega}^{k} T_{\omega}^{n\alpha} \Theta\right)(\tau,s)}{\chi(n,k)k!} (\omega - s)^{k+n\alpha}$$

$$= I_{\omega}^{(n-2)\alpha} \left((I_{\omega}^{m} T_{\omega}^{m}) T_{\omega}^{(n-2)\alpha} \Theta(\tau,\omega) \right) - \sum_{k=0}^{m-1} \times \frac{\left(T_{\omega}^{k} T_{\omega}^{(n-1)\alpha} \Theta\right)(\tau,s)}{\chi(n-1,k)k!} (\omega - s)^{k+(n-1)\alpha} - \sum_{k=0}^{m-1} \times \frac{\left(T_{\omega}^{k} T_{\omega}^{n\alpha} \Theta\right)(\tau,s)}{\chi(n,k)k!} (\omega - s)^{k+n\alpha}$$

$$= I_{\omega}^{(n-2)\alpha} \left(T_{\omega}^{(n-2)\alpha} \Theta(\tau,\omega) - \sum_{k=0}^{m-1} \frac{\left(T_{\omega}^{k} T_{\omega}^{(n-2)\alpha} \Theta\right)(\tau,s)}{k!} (\omega - s)^{k} \right) - \sum_{k=0}^{m-1} \times \frac{\left(T_{\omega}^{k} T_{\omega}^{(n-1)\alpha} \Theta\right)(\tau,s)}{\chi(n-1,k)k!} (\omega - s)^{k+(n-1)\alpha} - \sum_{k=0}^{m-1} \times \frac{\left(T_{\omega}^{k} T_{\omega}^{n\alpha} \Theta\right)(\tau,s)}{\chi(n,k)k!} (\omega - s)^{k+n\alpha}$$

If we keep the repeating of this process, then after n-times of scientific computations, we will obtain on the Eq. (2.27) which is the required. ■

Remark 2.2. The nth-partial sums of the CFTS (2.23) is given as

$$\Theta_n(\tau,\omega) = \sum_{r=0}^{n} \sum_{k=0}^{m-1} \frac{\left(T_{\omega}^{k} T_{\omega}^{r\alpha} \Theta\right)(\tau,s)}{\chi(r,k)k!} (\omega - s)^{k+r\alpha}. \tag{2.28}$$

So, by the previous theorem, the tail of the CFTS (the reminder of the series) is $\mathscr{R}_n(\tau,\omega) = I_{\omega}^{(n+1)\alpha} T_{\omega}^{(n+1)\alpha} \Theta(\tau,\omega)$. Thus, as any series, the CFTS is convergent when $\lim_{n\to\infty} \mathscr{R}_n(\tau,\omega) = 0$. The last theorem illustrates the required conditions for the convergence of the CFTS (2.23).

Theorem 2.3. *Let $\alpha \in (m-1,m]$ and $T_{\omega}^{r\alpha}\Theta(\tau,\omega)$ exist for $r = 0,1,2,\cdots,n+1$. If $\left|T_{\omega}^{(n+1)\alpha}\Theta(\tau,\omega)\right| \le B(\tau)$ on $s \le \omega \le d$, then the reminder $\mathscr{R}_n(\tau,\omega)$ of the CFTS satisfies*

$$|\mathscr{R}_n(\tau,\omega)| \le \frac{B(\tau)}{\chi(n+1,0)} (\omega - s)^{(n+1)\alpha}, s \le \omega \le d. \tag{2.29}$$

Proof. According to the assumption $\left|T_{\omega}^{(n+1)\alpha}\Theta(\tau,\omega)\right| \le B(\tau)$, we have

$$-B(\tau) \le T_{\omega}^{(n+1)\alpha}\Theta(\tau,\omega) \le B(\tau), s \le \omega \le d. \tag{2.30}$$

Appling the operator $I_{\omega}^{(n+1)\alpha}$ on Eq. (2.30) gives the following inequality

$$-B(\tau) \prod_{r=1}^{n+1} \frac{\Gamma(1+r\alpha-m)}{\Gamma(1+r\alpha)} (\omega - s)^{(n+1)\alpha} \le I_{\omega}^{(n+1)\alpha} T_{\omega}^{(n+1)\alpha} \Theta(\tau,\omega) \le B(\tau) \prod_{r=1}^{n+1} \frac{\Gamma(1+r\alpha-m)}{\Gamma(1+r\alpha)} (\omega - s)^{(n+1)\alpha}.$$

By the Remark 2.2 and Eq. (2.24), we complete the proof. ■

3. Construction of fractional series solution

In this section, we construct an FSS for the higher-order LCF-PDE in Eq. (1.1) subject to the non-homogeneous ordinary initial conditions in Eq. (1.2). The RPS technique is used to obtain the coefficients of the FSS. The main idea of the construction is assuming that the solution of Eq. (1.1) has a multivariable fractional PS expansion that suggested in Eq. (2.3) and then determinizing the coefficients of the series. Exact solution sometimes is gained if there are a pattern in the coefficients of the series solution otherwise an approximate and numerical solution is obtained.

Prior to construct the FSS of the IVP (1.1) and (1.2), we assume that the multivariable function $\phi(\tau,\omega)$ can be expanded in terms of CFMTS as follows:

$$\phi(\tau,\omega) = \sum_{r=0}^{\infty} \sum_{k=0}^{m-1} \frac{\left(T_{\omega}^{k} T_{\omega}^{r\alpha} \phi\right)(\tau,\omega_0)}{\chi(r,k)k!} (\omega - \omega_0)^{r\alpha+k}. \tag{3.1}$$

The RPS method is a powerful technique used to determine the coefficients of the PS solution. To illustrate the steps for determining these coefficients with the RPS method, we have formulated these steps in the following algorithm:

Step 1. Suppose that the solution of the IVP (1.1) and (1.2) has a multivariable fractional PS expansion about the initial point $\omega = \omega_0$ as follows:

$$\Theta(\tau,\omega) = \sum_{k=0}^{m-1} \frac{f_{0k}(\tau)}{k!} (\omega - \omega_0)^{k} + \sum_{r=1}^{\infty} \sum_{k=0}^{m-1} \times \frac{f_{rk}(\tau)}{\chi(r,k)k!} (\omega - \omega_0)^{r\alpha+k}, \tag{3.2}$$

where $\alpha \in (m-1,m], \tau \in I, \omega \in [\omega_0, \omega_0 + R), f_{0k}(\tau) = \left(T_{\omega}^{k}\Theta\right)(\tau,\omega_0)$, $f_{rk}(\tau) = \left(T_{\omega}^{k} T_{\omega}^{r\alpha}\Theta\right)(\tau,\omega_0)$ and R is the radius of convergence of the series.

The following step determines the first of m coefficients of the series in Eq. (3.2).

Step 2. Apply the operator $T_{\omega}^{j}, j = 0,1,2,\cdots,m-1$ on Eq. (3.2). According to Lemma 2.1 and Remark 2.1, we get

$$T_{\omega}^{j}\Theta(\tau,\omega) = \sum_{k=j}^{m-1} \frac{f_{0k}(\tau)}{(k-j)!} (\omega - \omega_0)^{k-j} + \sum_{r=1}^{\infty} \sum_{k=0}^{m-1} \times \frac{f_{rk}(\tau)\Gamma(r\alpha+k+1)}{\chi(r,k)\Gamma(r\alpha+k+1-j)k!} (\omega - \omega_0)^{r\alpha+k-j}. \tag{3.3}$$

By knowing $\Theta(\tau,\omega)$ satisfy the initial conditions of Eq. (1.2) and thus according to Eq. (3.3),we find that $f_{0k}(\tau) = \varphi_k(\tau), k = 0,1,2,\cdots,m-1$. So, the initial guess of $\Theta(\tau,\omega)$ is $\Theta_0(\tau,\omega) = \sum_{k=0}^{m-1} \frac{\varphi_k(\tau)}{k!} (\omega - \omega_0)^{k}$ and the multivariable fractional PS expansion in Eq. (3.2) becomes as

$$\Theta(\tau,\omega) = \sum_{k=0}^{m-1} \frac{\varphi_k(\tau)}{k!} (\omega - \omega_0)^{k} + \sum_{r=1}^{\infty} \sum_{k=0}^{m-1} \times \frac{f_{rk}(\tau)}{\chi(r,k)k!} (\omega - \omega_0)^{r\alpha+k}. \tag{3.4}$$

Step 3. Define the so-called residual functions of Eq. (1.1) as follows:

$$\mathrm{Res}(\tau,\omega) = T_{\omega}^{\alpha}\Theta(\tau,\omega) - \phi(\tau,\omega) - \sum_{j=0}^{m-1} \left(h_j(\tau)(\omega - \omega_0)^{j} T_{\omega}^{j} \left[L_{\tau}^{j} \Theta(\tau,\omega) \right] \right), \tau \in I, \omega \ge \omega_0, \tag{3.5}$$

Indeed, the name of RPS method comes from the residual function concept and the main idea of the method is based on the following facts about this function [29-32,42-44]:

(i) $\text{Res}(\tau,\omega) = 0$ for each $\omega \in [\omega_0, \omega_0 + R), \tau \in I$.
(ii) $T_\omega^{r\alpha}\text{Res}(\tau,\omega) = 0$, for each $r = 0, 1, 2, 3, \cdots$, according to part (1) of Lemma 2.1.
(iii) $T_\omega^k T_\omega^{r\alpha}\text{Res}(\tau,\omega_0) = 0$ for each $r = 0, 1, 2, 3, \cdots$ and $k = 0, 1, 2, \cdots, m-1$.

Step 4. Substitute Eq. (3.4) into Eq. (3.5), then we have

$$\text{Res}(\tau,\omega) = \sum_{k=0}^{m-1} \frac{f_{1k}(\tau)}{k!}(\omega-\omega_0)^k + \sum_{r=2}^{\infty}\sum_{k=0}^{m-1} \times \frac{f_{rk}(\tau)}{\chi(r-1,k)k!}(\omega-\omega_0)^{(r-1)\alpha+k}$$

$$-\sum_{k=0}^{m-1} \frac{\varphi_k(\tau)}{k!}(\omega-\omega_0)^k - \sum_{r=1}^{\infty}\sum_{k=0}^{m-1} \frac{f_{rk}(\tau)}{\chi(r,k)k!}(\omega-\omega_0)^{r\alpha+k}$$
$$-\sum_{j=0}^{m-1} h_j(\tau)\left(\sum_{k=j}^{m-1} \frac{L_\tau^j[\varphi_k(\tau)]}{(k-j)!}(\omega-\omega_0)^k + \sum_{r=1}^{\infty}\sum_{k=0}^{m-1} \frac{L_\tau^j[f_{rk}(\tau)]\Gamma(r\alpha+k+1)}{\chi(r,k)\Gamma(r\alpha+k+1-j)k!}(\omega-\omega_0)^{r\alpha+k}\right). \quad (3.6)$$

Step 5. Operate $T_\omega^i, i = 0, 1, \cdots, m-1$ on Eq. (3.6), then we obtain the following formula:

$$T_\omega^i\text{Res}(\tau,\omega) = \sum_{k=i}^{m-1} \frac{f_{1k}(\tau)}{(k-i)!}(\omega-\omega_0)^{k-i} + \sum_{r=2}^{\infty}\sum_{k=0}^{m-1} \times \frac{f_{rk}(\tau)\Gamma((r-1)\alpha+k+1)(\omega-\omega_0)^{(r-1)\alpha+k-i}}{\chi(r-1,k)\Gamma((r-1)\alpha+k-i+1)k!}$$

$$-\sum_{k=i}^{m-1} \frac{\varphi_k(\tau)}{(k-i)!}(\omega-\omega_0)^{k-i} - \sum_{r=1}^{\infty}\sum_{k=0}^{m-1} \frac{f_{rk}(\tau)\Gamma(r\alpha+k+1)}{\chi(r,k)\Gamma(r\alpha+k-i+1)k!}(\omega-\omega_0)^{r\alpha+k}$$
$$-\sum_{j=0}^{m-1} h_j(\tau)\left(\sum_{\substack{k=j, j\ge i\\ k=i, j<i}}^{m-1} \frac{L_\tau^j[\varphi_k(\tau)]k!}{(k-j)!(k-i)!}(\omega-\omega_0)^{k-i} + \sum_{r=1}^{\infty}\sum_{k=0}^{m-1} \frac{L_\tau^j[f_{rk}(\tau)](\Gamma(r\alpha+k+1))^2}{\chi(r,k)\Gamma(r\alpha+k+1-j)\Gamma(r\alpha+k-i+1)k!}(\omega-\omega_0)^{r\alpha+k}\right). \quad (3.7)$$

It is clear if we substitute $\omega = \omega_0$ in Eq. (3.7), then all the terms of degree greater than zero will be vanished. This fact will determine the first group of the unknown coefficients in Eq. (3.4) as we will see in the following step.

Step 6. By solving the algebraic equations $T_\omega^k\text{Res}(\tau,\omega_0) = 0$, $i = 0, 1, 2, \cdots, m-1$, we can obtain the form of the first group of the unknown coefficients in Eq. (3.4) which have the following form:

$$f_{1k}(\tau) = \left(T_\omega^k\phi\right)(\tau,\omega_0) + \sum_{j=0}^{k} h_j(\tau)\frac{L_\tau^j[\varphi_k(\tau)]\Gamma(k+1)}{\Gamma(k+1-j)}, k = 0, 1, 2, \cdots, m-1. \quad (3.8)$$

Step 7. Apply the operator T_ω^α on both sides of Eq. (3.6) to obtain

$$T_\omega^\alpha\text{Res}(\tau,\omega) = \sum_{k=0}^{m-1} \frac{f_{2k}(\tau)}{k!}(\omega-\omega_0)^k + \sum_{r=3}^{\infty}\sum_{k=0}^{m-1} \times \frac{f_{rk}(\tau)}{\chi(r-2,k)k!}(\omega-\omega_0)^{(r-2)\alpha+k}$$
$$-\sum_{k=0}^{m-1} \frac{f_{1k}(\tau)}{k!}(\omega-\omega_0)^k - \sum_{r=2}^{\infty}\sum_{k=0}^{m-1} \frac{f_{rk}(\tau)}{\chi(r-1,k)k!}(\omega-\omega_0)^{(r-1)\alpha+k}$$
$$-\sum_{j=0}^{m-1} h_j(\tau)\left(\sum_{k=0}^{m-1} \frac{L_\tau^j[f_{1k}(\tau)]\Gamma(\alpha+k+1)}{\Gamma(\alpha+k+1-j)k!}(\omega-\omega_0)^k + \sum_{r=2}^{i}\sum_{k=0}^{m-1} \frac{L_\tau^j[f_{rk}(\tau)]\Gamma(r\alpha+k+1)}{\chi(r-1,k)\Gamma(r\alpha+k+1-j)k!}(\omega-\omega_0)^{(r-1)\alpha+k}\right). \quad (3.9)$$

Step 8. Similar to Steps 5 and 6, operate $T_\omega^i, i = 0, 1, \cdots, m-1$ on Eq. (3.9) and solve the algebraic equations $T_\omega^k T_\omega^\alpha\text{Res}_i(\tau,\omega_0) = 0, k = 0, 1, 2, \cdots, m-1$, then the second group of the coefficients of Eq. (3.4) takes the following form:

$$f_{2k}(\tau) = \left(T_\omega^k T_\omega^\alpha\phi\right)(\tau,\omega_0) + \sum_{j=0}^{m-1} h_j(\tau)\frac{L_\tau^j[f_{1k}(\tau)](\tau)\Gamma(\alpha+k+1)}{\Gamma(\alpha+k+1-j)}, k = 0, 1, 2, \cdots, m-1. \quad (3.10)$$

Step 9. In general, to determine the r-group of coefficients of the series in Eq. (3.4), we need to solve the algebraic equation $T_\omega^k T_\omega^{(r-1)\alpha}\text{Res}(\tau,\omega_0) = 0, k = 0, 1, 2, \cdots, m-1, r = 1, 2, \cdots$. Hence, by some computation we can conclude that the general form of the coefficients of the multivariable FSS of the IVP (1.1) and (1.2) will be as follows:

$$f_{rk}(\tau) = \left(T_\omega^k T_\omega^{(r-1)\alpha}\phi\right)(\tau,\omega_0) + \sum_{j=0}^{m-1} h_j(\tau)\frac{L_\tau^j[f_{(r-1)k}(\tau)]\Gamma(\alpha+k+1)}{\Gamma(\alpha+k+1-j)}, k = 0, 1, \cdots, m-1, r = 2, 3, \cdots. \quad (3.11)$$

Step 10. The truncation of the series (3.4) represents an approximate solution of the IVP (1.1) and (1.2). For this, let $\Theta_i(\tau,\omega)$ denote to the ith-truncated series of the expansion in Eq. (3.4). That is

$$\Theta_i(\tau,\omega)=\sum_{k=0}^{m-1}\frac{\varphi_k(\tau)}{k!}(\omega-\omega_0)^k+\sum_{r=1}^{i}\sum_{k=0}^{m-1}\frac{f_{rk}(\tau)}{\chi(r,k)k!}(\omega-\omega_0)^{r\alpha+k},\quad i=1,2\cdots. \tag{3.12}$$

Like any series, if we increase the number of series terms, the error will be reduced. For this, calculation further coefficients of the FSS in Eq. (3.4) will give us a more accurate solution. Moreover, if there is a pattern in the series coefficients, the exact solution can be obtained as we shall see later.

4. Physical applications

Five applications of higher-order LCF-PDEs are carried out in this section to validate our proposed method. The time- conformable fractional wave equation is discussed in application 4.1, the space-conformable fractional telegraph equations is considered in applications 4.2 & 4.3, the time-fractional Poisson equation is considered in application 4.4, and the time-fractional Naiver-Stokes equation is considered in the last application 4.5. The MATHEMATICA 7 software package is used in our computational process.

It is worth noting that the equations in this section were originally modeled with Ca-FD (see Refs. [11-13]), but we replaced Co-FD with Ca-FD in order to compare the validity of using Co-FD instead of Ca- FD in constructing fractional differential equations for the advantages of definition Co-FD over definition Ca-FD. The most important advantages of definition Co-FD from the definition of Ca-FD are that it simulates the most concepts and properties of an ordinary derivative such as: quotient, product and chain rules while the Ca-FD definition fail to meet these rules, a non-differentiable function in Caputo sense can be differentiated in the conformable sense and so the size of computations in the Co-FD are less than in the Ca-FD.

Application 4.1: Homogeneous time-conformable fractional wave equation

$$T_{\omega}^{\alpha}\Theta(\tau,\omega)=\frac{1}{2}\tau^2\frac{\partial^2\Theta(\tau,\omega)}{\partial\tau^2},\tau\in\mathbb{R},\omega\geq 0,1<\alpha\leq 2, \tag{4.1}$$

with ordinary initial conditions

$$\Theta(\tau,0)=\tau,\Theta_{\omega}(\tau,0)=\tau^2. \tag{4.2}$$

The comparison of the IVP (4.1) and (4.2) with the IVP (1.1) and (1.2) shows us that $m=2$, $\omega_0=0$, $\varphi_0(\tau)=\tau$ and $\varphi_1(\tau)=\tau^2$, $\phi(\tau,\omega)=0$, $h_0(\tau)=\frac{1}{2}\tau^2$, $h_1(\tau)=0$,$L_{\tau}^{0}=\frac{\partial^2}{\partial\tau^2}$. Thus, according to Eq. (3.4), the FSS of the IVP (4.1) and (4.2) will be as follows:

$$\Theta(\tau,\omega)=\tau+\tau^2\omega+\sum_{r=1}^{\infty}\sum_{k=0}^{1}\frac{f_{rk}(\tau)\omega^{r\alpha+k}}{\chi(r,k)}. \tag{4.3}$$

According to Eqs. (3.8), (3.10) and (3.11), the first, second and third groups of the coefficients of the FSS in Eq. (4.3), have the following forms, respectively:

$$f_{10}(\tau)=0,f_{11}(\tau)=\frac{\tau^2\chi(1,1)}{\alpha(1+\alpha)}, \tag{4.4}$$

$$f_{20}(\tau)=0,f_{21}(\tau)=\frac{\tau^2\chi(2,1)}{2\alpha^2(1+\alpha)(1+2\alpha)}, \tag{4.5}$$

$$f_{30}(\tau)=0,f_{31}(\tau)=\frac{\tau^2\chi(3,1)}{6\alpha^3(1+\alpha)(1+2\alpha)(1+3\alpha)}. \tag{4.6}$$

Therefore, the 3rd-approximate solution (3rd-truncated series) of the IVP (4.1) and (4.2) can be expressed as:

$$\Theta_3(\tau,\omega)=\tau+\tau^2\left(\omega+\frac{\omega^{1+\alpha}}{\alpha(1+\alpha)}+\frac{\omega^{1+2\alpha}}{2\alpha^2(1+\alpha)(1+2\alpha)}+\frac{\omega^{1+3\alpha}}{6\alpha^3(1+\alpha)(1+2\alpha)(1+3\alpha)}\right). \tag{4.7}$$

It can be noted that the pattern between the terms of the series, and therefore the exact FSS of the IVP (4.1) and (4.2) takes the following form:

$$\Theta(\tau,\omega)=\tau+\tau^2\sum_{r=0}^{\infty}\frac{\omega^{1+r\alpha}}{r!\alpha^r\prod_{j=0}^{r}(1+j\alpha)}. \tag{4.8}$$

For $\alpha=2$, the solution in Eq. (4.8) becomes:

$$\Theta(\tau,\omega)=\tau+\tau^2\left(\omega+\frac{\omega^3}{3!}+\frac{\omega^5}{5!}+\frac{\omega^7}{7!}+\cdots\right). \tag{4.9}$$

So, the exact solution of the IVP. (4.1) and (4.2) in a closed form will be $\Theta(\tau,\omega)=\tau+\tau^2\sinh\omega$.

In ref. [11], the authors considered the fractional derivative, in Eq. (4.1), in Ca-FD sense and got an approximate solution for IVP (4.1) and (4.2) by Adomian decomposition method (ADM). The 3rd-approximate ADM solution was as follows:

$$\Theta_3(\tau,\omega)=\tau+\tau^2\left(\omega+\frac{\omega^{1+\alpha}}{\Gamma(2+\alpha)}+\frac{\omega^{1+2\alpha}}{\Gamma(2+2\alpha)}+\frac{\omega^{1+3\alpha}}{\Gamma(2+3\alpha)}\right). \tag{4.10}$$

which simulates the approximate solution in Eq. (4.7) in its terms.

Anyhow, Fig. 1 shows the surface graphs of the approximate solutions of the IVP (4.1) and (4.2) in sense of Co-FD for deferent values of α whereas Fig. 2 shows the approximate solutions of the IVP (4.1) and (4.2) in sense of Ca-FD. It is clear from the figures that the behavior of the graphs in the first case is similar to their behavior in the second case, so it is easy to say that modeling wave problems using the Co-FD can be an alternative to using Ca-FD.

To highlight more comparisons between the solutions of the IVP (4.1) and (4.2) in both cases the Co-FD and Ca-FD, we plotted the solutions at $\tau=1$ and for different values of α. Fig. 3(a) and (b) confirm that the solution of the IVP (4.1) and (4.2) in both cases has the same behavior, however in the case of Ca-FD the solution is slightly faster to increase than in the case Co-FD. Furthermore, note that the solution reduces its growth as α approaches 2 in both cases, Co-FD and Ca-FD.

To measure the accuracy of the approximate solution we obtained in Eq. (4.7) and compare with the solution in Eq. (4.10), we plot in Fig. 4 the relative error at $\tau=1$ for different values of α. We define the relative error in the following formula:

$$Relative\ \ Err(\tau,\omega)=\left|\frac{\Theta_{10}(\tau,\omega)-\Theta_3(\tau,\omega)}{\Theta_{10}(\tau,\omega)}\right|.$$

Fig. 4 shows that the relative error of the 3rd-approximate solution of the IVP (4.1) and (4.2) in case of Co-FD is very small and better than the relative error of the 3rd-approximate solution in case of Ca-FD.

Application 4.2: Homogeneous space-conformable fractional telegraph equation:

$$T_{\omega}^{\alpha}\Theta(\tau,\omega)=\frac{\partial^2\Theta(\tau,\omega)}{\partial\tau^2}+\frac{\partial\Theta(\tau,\omega)}{\partial\tau}+\Theta(\tau,\omega),\tau\geq\mathbb{R},\omega\geq 0,1<\alpha\leq 2, \tag{4.11}$$

with nonhomogeneous initial conditions

$$\Theta(\tau,0)=\Theta_{\omega}(\tau,0)=e^{-\tau}, \tag{4.12}$$

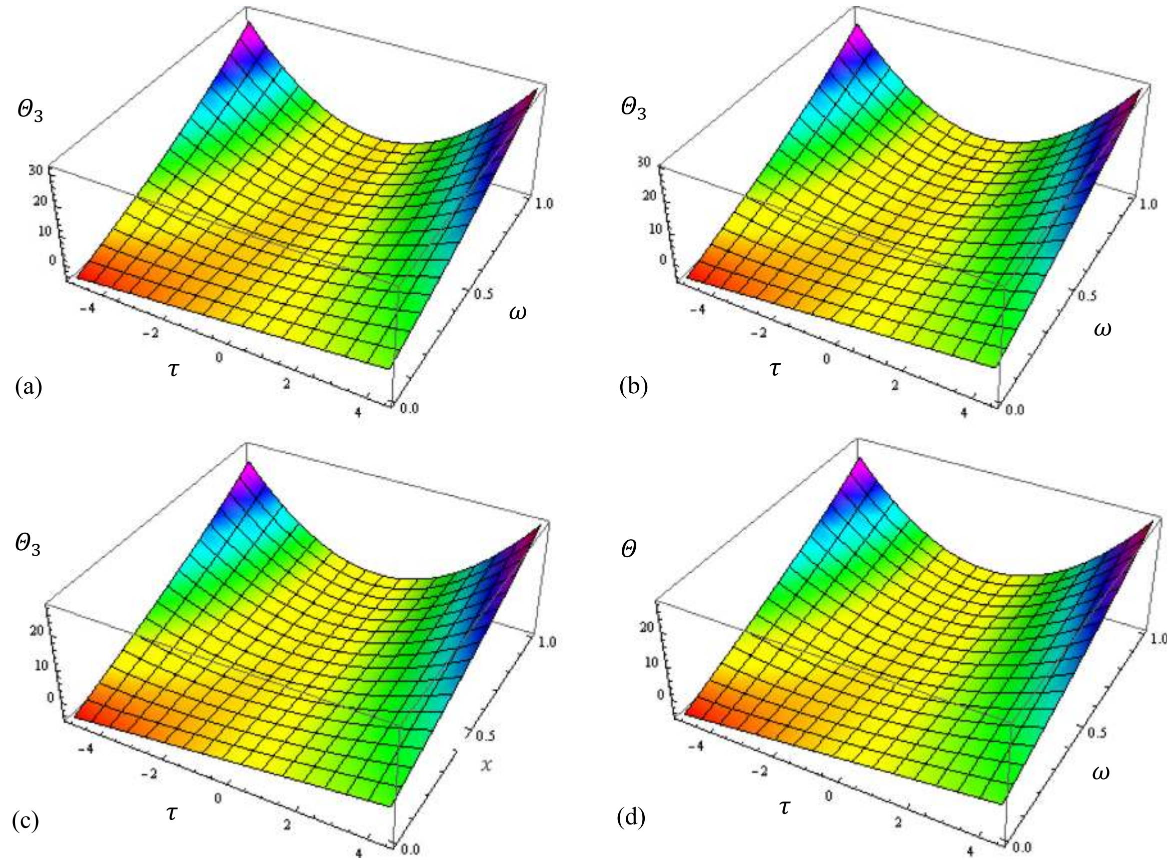

Fig. 1. The surface graphs of the approximate solutions (Eq. (4.7)) of the IVP (4.1) and (4.2): (a) $\Theta_3(\tau,\omega)$ when $\alpha = 4/3$, (b) $\Theta_3(\tau,\omega)$ when $\alpha = 5/3$, (c) $\Theta_3(\tau,\omega)$ when $\alpha = 2$, (d) $\Theta = \tau + \tau^2 \sinh\omega$ (the exact solution).

where α is a parameter describing the order of the space-conformable fractional derivative and $\Theta(\tau,\omega)$ is assumed to be a causal function of space, i.e., vanishing for $\omega < 0$.

Refer to the IVP (1.1) and (1.2) and compare it with the IVP (4.11) and (4.12), we find that $m = 2$, $\omega_0 = 0$, $\varphi_0(\tau) = e^{-\tau}$ and $\varphi_1(\tau) = e^{-\tau}$, $\phi(\tau,\omega) = 0$, $h_0(\tau) = 1$, $h_1(\tau) = 0$, $L_\tau^0 = \frac{\partial^2}{\partial\tau^2} + \frac{\partial}{\partial\tau} + I$ where I is the identity differential operator. Therefore, depending on Eq. (3.4), the FSS of the IVP (4.11) and (4.12) will be as follows:

$$\Theta(\tau,\omega) = e^{-\tau} + e^{-\tau}\omega + \sum_{r=1}^{\infty}\sum_{k=0}^{1} \frac{f_{rk}(\tau)\omega^{r\alpha+k}}{\chi(r,k)}, \tag{4.13}$$

and according to Eq. (3.11), $L_\tau^0\left[f_{(r-1)k}(\tau)\right] = e^{-\tau}$ and so the coefficients $f_{rk}(\tau) = e^{-\tau}, r = 1,2,\cdots,k = 0,1$. Hence, the exact FSS of the IVP (4.11) and (4.12) will be of the following form:

$$\Theta(\tau,\omega) = e^{-\tau}\sum_{r=0}^{\infty}\sum_{k=0}^{1} \frac{\omega^{r\alpha+k}}{\chi(r,k)}. \tag{4.14}$$

The ADM solution for the IVP (4.11) and (4.12) [11], where the fractional derivative was considered in Ca-FD sense, was given in the following infinite series:

$$\Theta(\tau,\omega) = e^{-\tau}\left(1+\omega+\frac{\omega^\alpha}{\Gamma(1+\alpha)}+\frac{\omega^{1+\alpha}}{\Gamma(2+\alpha)}+\frac{\omega^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\omega^{1+2\alpha}}{\Gamma(2+2\alpha)}+\cdots\right). \tag{4.15}$$

As a special case when $\alpha = 2$, the solutions in (4.14) and (4.15) tend to the following infinite series

$$\Theta(\tau,\omega) = e^{-t}\left(1+\omega+\frac{\omega^2}{2!}+\frac{\omega^3}{3!}+\frac{\omega^4}{4!}+\frac{\omega^5}{5!}+\cdots\right). \tag{4.16}$$

which equivalent to the closed exact solution $\Theta(\tau,\omega) = e^{\omega-\tau}$. It should be noted that the solution of the IVP (4.1) and (4.2) when the value of α is very close to the number 2, the solution, whether in the case of Co-FD or case Ca-FD will approach the solution in the case of the use of the ordinary second derivative.

Anyhow, Table 1 illustrates some comparisons between the solutions of the IVP (4.11) and (4.12) when the fractional derivative is considered in the conformable sense (the solution in Eq. (4.14)) and when it is considered in the sense of Ca-FD (the solution in Eq. (4.15)) for $\alpha = 1.5$. The third and fourth columns represent the values of the 3rd-approximation, $\Theta_3(\tau,\omega)$, of the solutions in Eqs. (4.14) and (4.15), respectively, at various points in the region $[0,\infty)\times[0,1]$ that given in the first and second columns. The fifth and sixth columns represents the values of the residual error (Re. Err) for the 3rd-approximation of the solutions in Eqs. (4.14) and (4.15), respectively, where the residual error of $\Theta_3(\tau,\omega)$ of the Eq. (4.11) is defined by

$$\text{Re.Err}(\tau,\omega) = \left|\frac{\partial^\alpha\Theta_3(\tau,\omega)}{\partial\omega^\alpha} - \frac{\partial^2\Theta_3(\tau,\omega)}{\partial\tau^2} - \frac{\partial\Theta_3(\tau,\omega)}{\partial\tau} - \Theta_3(\tau,\omega)\right|. \tag{4.17}$$

Further, comparison of the data in columns 3 and 4 shows that the solutions in Eqs. (4.14) and (4.15) are very similar and that the difference between the data increases by increasing the value of ω, while the difference decreases by stabilizing the value of ω and increasing the value of τ. Anyway, the data shows that the behavior

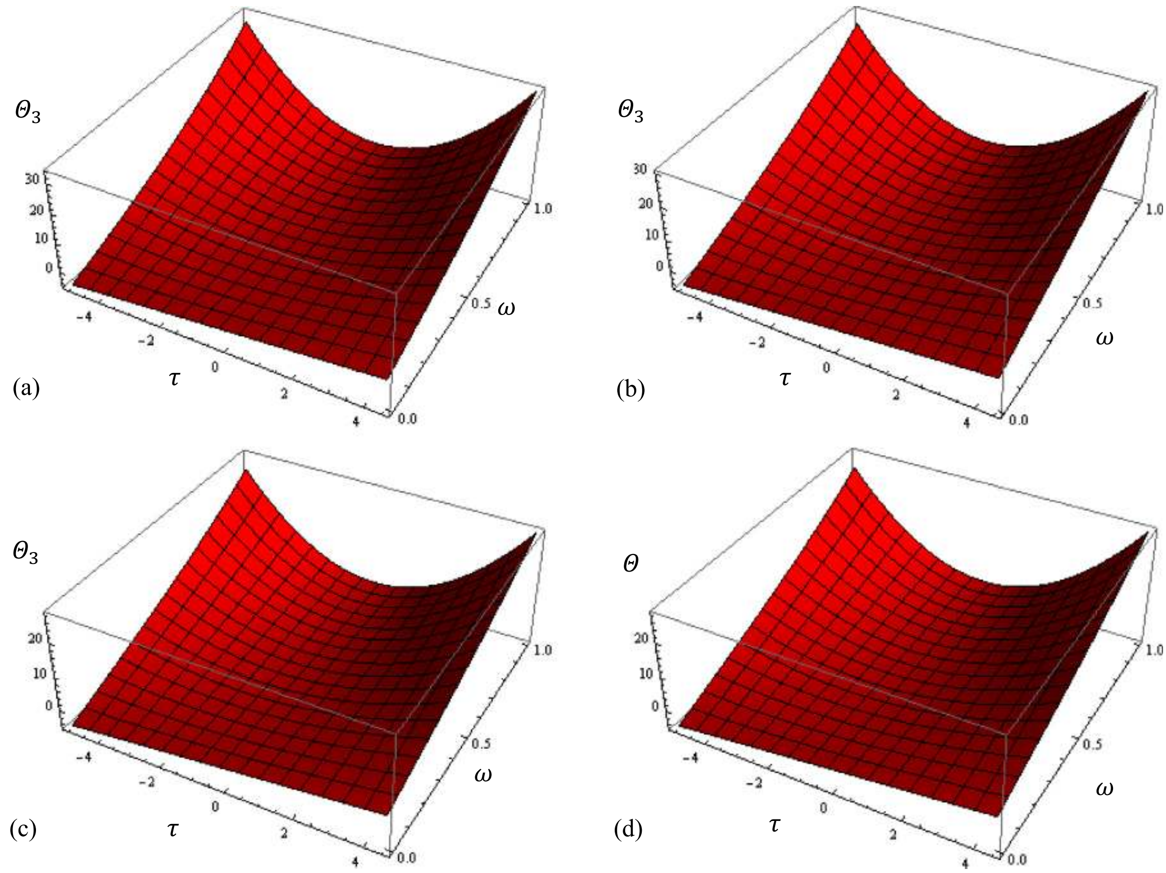

Fig. 2. The surface graphs of the approximate solutions (Eq. (4.10)) of the IVP (4.1) and (4.2): (a) $\Theta_3(\tau,\omega)$ when $\alpha = 4/3$, (b) $\Theta_3(\tau,\omega)$ when $\alpha = 5/3$, (c) $\Theta_3(\tau,\omega)$ when $\alpha = 2$, (d) $\Theta = \tau + \tau^2 \sinh\omega$ (the exact solution).

of the solutions in Eqs. (4.14) and (4.15) are similar, regardless of the change in the values of ω and τ increasing or decreasing. What is striking, however, is that the remaining error of the solution in Eq. (4.14) is slightly lower than that of the solution in Eq. (4.15). This indicates that computations when using Co-FD are smoother and easier than the case in which Ca-FD is used.

Application 4.3: Nonhomogeneous space-conformable fractional telegraph equation:

$$T^{\alpha}_{\omega}\Theta(\tau,\omega) = \frac{\partial^2\Theta(\tau,\omega)}{\partial\tau^2} + \frac{\partial\Theta(\tau,\omega)}{\partial\tau} + \frac{\omega^2}{2} - 3\omega - 1, \tau \geq 0, \omega \geq 0, 2 < \alpha \leq 3, \tag{4.18}$$

with nonhomogeneous ordinary initial conditions

$$\Theta(\tau,0) = \tau, \Theta_{\omega}(\tau,0) = 0, \Theta_{\omega\omega}(\tau,0) = 2 - \tau. \tag{4.19}$$

By comparing the IVP (4.18) and (4.19) with the IVP (1.1) and (1.2), we find $m = 3$, $\omega_0 = 0$, $\varphi_0(\tau) = \tau, \varphi_1(\tau) = 0$ and $\varphi_2(\tau) = 2 - \tau$, $\phi(\tau,\omega) = \frac{\omega^2}{2} - 3\omega - 1$, $h_0(\tau) = 1$, $h_1(\tau) = h_2(\tau) = 0$, $L^0_{\tau} = \frac{\partial^2}{\partial\tau^2} + \frac{\partial}{\partial\tau}$. Thus, according to Eq. (3.4), the FSS of IVP (4.18) and (4.19) has the following form:

$$\Theta(\tau,\omega) = \tau + \frac{(2-\tau)\omega^2}{2} + \sum_{r=1}^{\infty}\sum_{k=0}^{2}\frac{f_{rk}(\tau)}{\chi(r,k)k!}\omega^{r\alpha+k}. \tag{4.20}$$

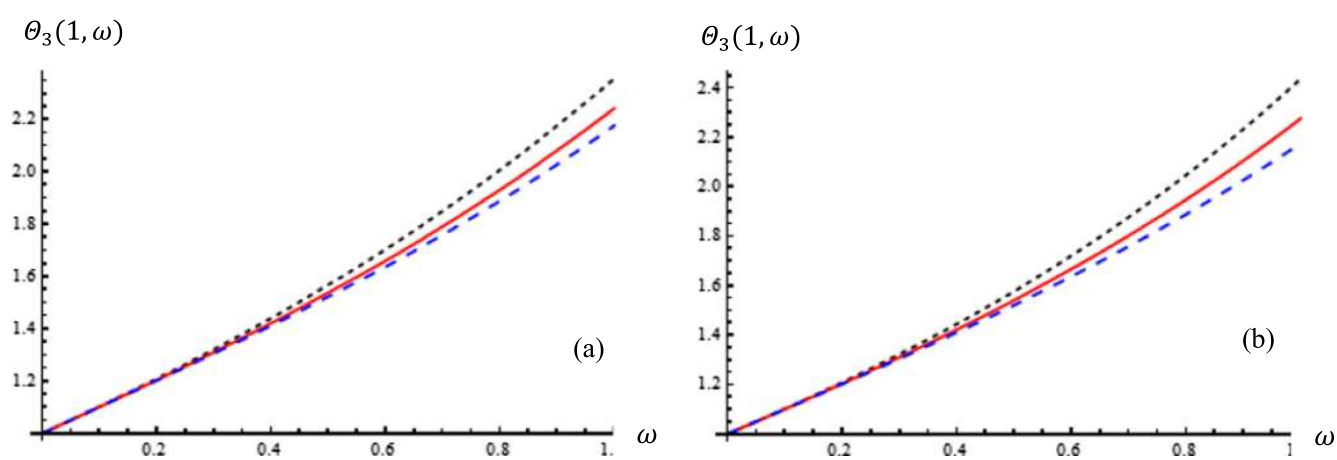

Fig. 3. The approximate solutions of the IVP (4.1) and (4.2) at $\tau = 1$; $\Theta_3(1,\omega)$ when $\alpha = 4/3$ (Dotted curve), $\alpha = 5/3$ (Solid curve) and $\alpha = 2$ (Dashed curve): (a) in the sense of Co-FD (Eq. (4.7)), (b) in the sense of Ca-FD (Eq. (4.10)).

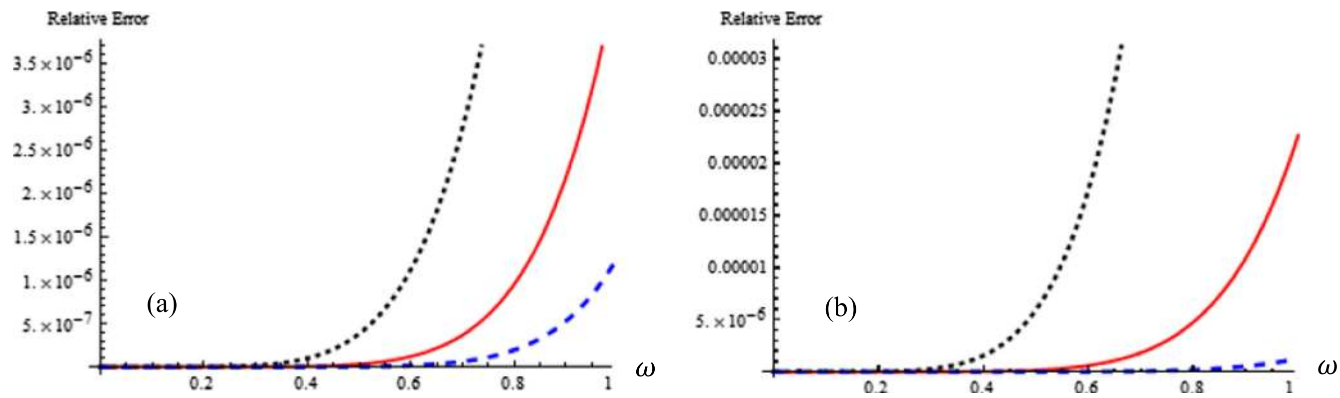

Fig. 4. The relative error of the approximate solutions of the IVP (4.1) and (4.2) at $\tau = 1$; when $\alpha = 4/3$ (Dotted curve), $\alpha = 5/3$ (Solid curve) and $\alpha = 2$ (Dashed curve): (a) in the sense of Co-FD (Eq. (4.7)), (b) in the sense of Ca-FD (Eq. (4.10)).

To determine the form of the first group of the coefficients in Eq. (4.20), $f_{1k}(\tau)$, for $k = 0, 1, 2$, we need to use Eq. (3.8). Thus, $f_{10}(\tau) = 0, f_{11}(\tau) = -3$, and $f_{12}(\tau) = 0$. So, the Eq. (4.20) becomes

$$\Theta(\tau,\omega) = \tau + \frac{(2-\tau)\omega^2}{2} - \frac{3\omega^{\alpha+1}}{\alpha^3-\alpha} + \sum_{r=2}^{\infty}\sum_{k=0}^{2}\frac{f_{rk}(\tau)}{\chi(r,k)k!}\omega^{r\alpha+k}. \quad (4.21)$$

Refer to Eq. (3.10) and (3.11), we find that the rest of the coefficients vanish. Hence, we have obtained the exact solution for IVP (4.18) and (4.19):

$$\Theta(\tau,\omega) = \tau + \frac{(2-\tau)\omega^2}{2} - \frac{3\omega^{\alpha+1}}{\alpha^3-\alpha}. \quad (4.22)$$

However, in the ref. [11], the authors presented a solution to the IVP (4.18) and (4.19), where it was the fractional derivative in the Ca-FD concept, then the exact solution was as follows:

$$\Theta(\tau,\omega) = \tau + \frac{(2-\tau)\omega^2}{2} - \frac{3\omega^{\alpha+1}}{\Gamma(2+\alpha)}. \quad (4.23)$$

From the similarity between the solution presented in Eq. (4.22) and Eq. (4.23), we find that the difference in part of the coefficients while the number and form of the terms are equal and similar.

The solution to the telegraph equation, $\Theta(\tau,\omega)$, sometimes represents the voltage of the electric current at any point and any time. Table 2 shows the value of $\Theta(\tau,\omega)$ at several points in different times and for different values of parameter α. Data indicate that voltages increase with an increase in the values of α and this means that the amount of voltage can be controlled through the order of the Co-FD.

Application 4.4: Nonhomogeneous time-conformable fractional Poisson equation:

$$T_\omega^\alpha\Theta(\tau,\omega) = \frac{\partial^2\Theta(\tau,\omega)}{\partial\tau^2} + \omega\frac{\partial^2\Theta(\tau,\omega)}{\partial\omega\partial\tau} + \Theta(\tau,\omega) - \tau\omega, \tau \in \mathbb{R}, \omega \geq 0, 1 < \alpha \leq 2, \quad (4.24)$$

with nonhomogeneous ordinary initial conditions

$$\Theta(\tau,0) = 1, \Theta_\omega(\tau,0) = \tau. \quad (4.25)$$

Comparing the IVP (4.24) and (4.25) with the IVP (1.1) and (1.2) shows that $m = 2$, $\omega_0 = 0$, $\varphi_0(\tau) = 1, \varphi_1(\tau) = \tau$, $\phi(\tau,\omega) = -\tau\omega$, $h_0(\tau) = h_1(\tau) = 1$, $L_\tau^0 = \frac{\partial^2}{\partial\tau^2} + I$, where I is the identity operator, and $L_\tau^1 = \frac{\partial}{\partial\tau}$. According to Eq. (3.4), the FSS of the IVP (4.24) and (4.25) will be as follows:

$$\Theta(\tau,\omega) = 1 + \tau\omega + \sum_{r=1}^{\infty}\sum_{k=0}^{1}\frac{f_{rk}(\tau)}{\chi(r,k)}\omega^{r\alpha+k}. \quad (4.26)$$

It is easy to verify, using Eqs. (3.8) and (3.11), that $f_{rk}(\tau) = 1$, for $r = 1, 2, \cdots$, and $k = 0, 1$. Therefore, the exact series solution for the IVP (4.24) and (4.25) is

$$\Theta(\tau,\omega) = 1 + \tau\omega + \sum_{r=1}^{\infty}\sum_{k=0}^{1}\frac{\omega^{r\alpha+k}}{\chi(r,k)}. \quad (4.27)$$

The authors in [13], introduced a series solution for the IVP (4.21) and (4.22), where the fractional derivative was in Ca-FD sense, which was of the form:

$$\Theta(\tau,\omega) = 1 + \tau\omega + \sum_{r=1}^{\infty}\sum_{k=0}^{1}\frac{\omega^{r\alpha+k}}{\Gamma(1+k+r\alpha)}. \quad (4.28)$$

Table 1
Numerical comparisons between the 3rd-approximation of the solutions in Eqs. (4.14) and (4.15) at $\alpha = 1.5$.

ω	τ	$\Theta_3(\tau,\omega)$- Ca-FD	$\Theta_3(\tau,\omega)$- Co-FD	Re.Err - Ca-FD	Re.Err - Co-FD
0.1	5	7.57959×10^{-3}	7.70303×10^{-3}	2.44715×10^{-2}	2.60198×10^{-2}
	10	5.10709×10^{-5}	5.19026×10^{-5}	1.64888×10^{-4}	1.27248×10^{-4}
	15	3.44113×10^{-7}	3.49717×10^{-7}	1.11101×10^{-6}	6.08842×10^{-7}
0.4	5	1.09996×10^{-2}	1.19874×10^{-2}	2.75721×10^{-2}	2.11332×10^{-2}
	10	7.41149×10^{-5}	8.07704×10^{-5}	1.85780×10^{-4}	6.75841×10^{-5}
	15	4.99382×10^{-7}	5.44226×10^{-7}	1.25177×10^{-6}	6.85933×10^{-8}
0.7	5	1.57068×10^{-2}	1.80022×10^{-2}	3.58282×10^{-2}	2.16579×10^{-2}
	10	1.05832×10^{-4}	1.21298×10^{-4}	2.41409×10^{-4}	3.35828×10^{-5}
	15	7.13087×10^{-7}	8.17299×10^{-7}	1.62660×10^{-6}	3.54580×10^{-7}
1.0	5	2.19757×10^{-2}	2.59037×10^{-2}	4.68166×10^{-2}	2.20346×10^{-2}
	10	1.48071×10^{-4}	1.74538×10^{-4}	3.15448×10^{-4}	1.31902×10^{-5}
	15	9.97698×10^{-7}	1.17602×10^{-6}	2.12547×10^{-6}	9.24682×10^{-7}

Table 2
Numerical results of the exact solution in Eq. (4.22) at different values of α.

ω	τ	$\alpha = 7/3$ $\Theta(\tau,\omega)$	$\alpha = 8/3$ $\Theta(\tau,\omega)$	$\alpha = 3$ $\Theta(\tau,\omega)$
0.1	0	0.01987	0.01996	0.01999
	2	1.99987	1.99996	1.99999
	4	4.01987	4.01996	4.01999
0.4	0	0.30636	0.31360	0.31680
	2	1.98636	1.99360	1.99680
	4	4.30636	4.31360	4.31680
0.7	0	0.89190	0.93022	0.94999
	2	1.91190	1.95022	1.96999
	4	4.89190	4.93022	4.94999
1.0	0	1.71071	1.81591	1.87500
	2	1.71071	1.81591	1.87500
	4	5.71071	5.81591	5.87500

Note that we can make one of the solutions in Eqs. (4.27) and (4.28) coincident to the other by multiplying the general term of the series by the factor $\frac{\chi(r,k)}{\Gamma(1+k+r\alpha)}$ or its reciprocal. On the other hand, as a special case when $\alpha = 2$, the solutions in Eqs. (4.27) and (4.28) can be expressed in the closed form $\Theta(\tau,\omega) = e^{\omega} - \omega + \tau\omega$.

Application 4.5: Nonhomogeneous time-fractional Navier-Stokes equation:

$$T^{\alpha}_{\omega}\Theta(\tau,\omega) = \lambda + \frac{\partial^2\Theta(\tau,\omega)}{\partial\tau^2} + \frac{1}{\tau}\frac{\partial\Theta(\tau,\omega)}{\partial\tau} = 0, \lambda \in \mathbb{R}, \tau > 0, \omega \geq 0, 0 < \alpha \leq 1, \tag{4.29}$$

with nonhomogeneous ordinary initial condition

$$\Theta(\tau,0) = 1 - \tau^2. \tag{4.30}$$

In this application $\alpha \in (0,1]$; that is $m = 1$ and thus $\chi(r,0) = a^r r!$. Therefore, the solution of the IVP (4.29) and (4.30) take the following expansion:

$$\Theta(\tau,\omega) = \sum_{r=0}^{\infty} f_r(\tau)\frac{\omega^{r\alpha}}{a^r r!}. \tag{4.31}$$

Depending on the standard form of the IVP (1.1) and (1.2), we can determine the basic tools to the solution of the IVP (4.29) and (4.30), which are $\omega_0 = 0$, $\varphi_0(\tau) = 1 - \tau^2$, $\phi(\tau,\omega) = p$, $h_0(\tau) = 1$, $L^0_{\tau} = \frac{\partial^2}{\partial\tau^2} + \frac{1}{\tau}\frac{\partial}{\partial\tau}$. According to Eqs. (3.8) and (3.11), we find $f_1(\tau) = \lambda - 4$ and $f_r(\tau) = 0$ for $= 2, 3, \cdots$. Therefore, the exact solution of the IVP (4.29) and (4.30) will be as follows:

$$\Theta(\tau,\omega) = 1 - \tau^2 + (\lambda - 4)\frac{\omega^{\alpha}}{\alpha}. \tag{4.32}$$

The IVP (4.29) and (4.30) has been resolved in the sense of Ca-FD in Ref. [12] and was of the following form:

$$\Theta(\tau,\omega) = 1 - \tau^2 + (\lambda - 4)\frac{\omega^{\alpha}}{\Gamma(1+\alpha)}, \tag{4.33}$$

and when $\alpha = 1$, the solutions in Eqs. (4.32) and (4.33) turn into the following polynomial $\Theta(\tau,\omega) = 1 - \tau^2 + (\lambda - 4)\omega$.

5. Conclusion

Due to the many advantages of the Co-FD, we have tried to add other features to this type of derivative. First, we were able to develop an appropriate expansion to solve a class of higher-order LCF-PDEs with non-homogeneous ordinary initial conditions. We offered an FSS for this kind of IVP by using the proposed expansion. The comparisons in the two cases, the use of Co-FD or the use of Ca-FD in high-order LF-PDEs, indicate that the behavior of the solution graphs is similar in both cases but with a slight difference of their local values that increase and decrease according to the values of independent variables. The comparisons in the two cases, the use of Co-FD or the use of Ca-FD in high-order LF-PDEs, indicate that the behavior of the solution graphs is similar in both cases but with a slight difference of their local values that increase sometimes and decrease at other times according to the values of independent variables. Therefore, with the many advantages of the Co-FD definition, the use of Co-FD in PDEs modeling can be an appropriate substitute for the Ca-FD and other types of fractional derivatives. It is important to mention that the performance of the mathematical and numerical operations to solve the LCF-PDE using Mathematica software did not exceed a few minutes compared to the time required for the equations themselves when using Ca-FD and this is another feature added to Co-FD. We observed that the RPS method is very suitable, easy and effective to solve such a class of PDEs and can be used to solve other types of differential equations. The expansion that is developed to create FSS for this class of linear PDEs is successful and consistent with the form of the target equations, but the question remains open to what extent this extension is suitable to construct FSS for other forms of linear and nonlinear PDEs?

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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