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SINGULAR INNER FUNCTION AND ANALYTIC BOUNDED POINT EVALUATION

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Abstract: In Section 1, I constructed a singular inner function which is an upper bound for a given outer function, i.e. replacing an outer function with a constant multiple of a singular inner

In Section 2, I proved how unit disk containment is not effected for the analytic bounded point evaluation by a restricted measure.

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1. Construction of a Singular Inner Function from a Given Outer One

Lets start first by introducing the following three definitions :

Definition 1.1. A function F defined and analytic in D is called an outer function if it has the form

$$F(z) = \chi exp[\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log k(t)dt] \qquad (z \in D),$$

where k is a non-negative Lebesgue measurable function on ∂D , $log(k(t)) \in L^1(m)$ and χ is a complex number of modulus 1

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Definition 1.2. A function S_μ defined and analytic in D is called a singular inner function if it has the form

$$S_\mu(z) = exp[-\int_{\partial D} \frac{\xi + z}{\xi - z} d\mu(z)] \qquad (z \in D),$$

where μ is a finite non-negative Borel measure on ∂D that is singular with respect to m

Definition 1.3. A subset E of ∂D is called a Carleson set if:

1) E is closed;

2) E is of Lebesgue measure zero ; $|E| = 0$;

3) $\sum_\nu |I_\nu| \log \frac{1}{|I_\nu|} < \infty$, where $\{I_\nu\}$ are the complementary arcs of E

Now, the following Theorem which appears in [1] is of great importance for the Theorem that follows

Theorem 1.4. *Let Ω be any Jordan domain in D, with $m(\partial D \bigcap \partial\Omega) = 0$, let $0 \leq h \in L^1(dm)$ and let $P_z(\zeta)$ be the Poisson kernal on ∂D evaluated at z in Ω . Then, for any $\epsilon > 0$, there exists a finite measure μ with support in ∂D such that $\mu \perp m$, and $\mu(E) = 0$ for every Carleson set E in ∂D and $|\int_{\partial D}(P_z(\zeta)d\mu - P_z(\zeta)hdm)| \leq \epsilon$ for all z in Ω.*

In this Theorem we construct a special singular inner function which is an upper bound for a given outer function

Theorem 1.5. *Let Ω be any Jordan domain in D , with $m(\partial D \bigcap \partial\Omega) = 0$ and let F be an outer function on D; $F \neq 0$. Then there exists a singular inner function S_μ, with $\mu(E) = 0$ for every Carleson set E in ∂D, such that $|S_\mu(z)| \leq 3.|F(z)|$ for all z in Ω.*

Proof. Since F is an outer function, $|F(z)| = e^{\int_{\partial D} P_z(\zeta) \log h(\zeta) dm(\zeta)}$ without loss of generality we can assume that $0 \leq h \leq 1$ on ∂D. Now, by Theorem 1.4 , there exists a finite positive Borel measure μ such that $\mu \perp m$, $\mu(E) = 0$ for every Carleson set $E \subseteq \partial D$ and $|\int_{\partial D}(P_z(\zeta)d\mu(\zeta) - P_z(\zeta)) \log h(\zeta) dm(\zeta))| \leq 1$ for all z in Ω. Let S_μ be the corresponding singular inner function. Then $|S_\mu(z)| = e^{-\int_{\partial D} P_z(\zeta) d\mu(\zeta)}$ and therefore, $|S_\mu(z)/F(z)| = e^{-\int_{\partial D}(P_z(\zeta)d\mu - \int_{\partial D} P_z(\zeta)\log(h) dm)} \leq e^{\int_{\partial D}(P_z(\zeta)|log(h)|dm - P_z(\zeta)d\mu)} \leq 3$ for all z in Ω. □

2. The set of Overconvergence $abpe(P^t(\mu))$

We now turn to the topic of point evaluations and establish some "overconvergence" results

With μ and $P^t(\mu)$ as before, a complex number z is called a bounded point evaluation for $P^t(\mu)$ if there is a constant M such that $|p(z)| \leq M.||p||_{L^t(\mu)}$ for all polynomials p; the collection of all such points is denoted $bpe(P^t(\mu))$. If $z \in C$ and there are positive constants M and r such that $|p(w)| \leq M.||p||_{L^t(\mu)}$ whenever p is a polynomial and $|w - z| < r$, then we call z an analytic bounded point evaluation for $P^t(\mu)$; the set of all points z of this type is denoted by $abpe(P^t(\mu))$. Notice that $abpe(P^t(\mu))$ is an open subset of $bpe(P^t(\mu))$ and, by the Maximum Modulus Theorem, each component of $abpe(P^t(\mu))$ is simply connected. If $z \in abpe(P^t(\mu))$, then by the Hahn-Banach and Riesz Representation Theorems, there exists K, in $L^s(\mu)$ $(1/s + 1/t = 1)$ such that $p(z) = \int p(\xi)K_z(\xi)d\mu(\xi)$ for each polynomial p. For f in $P^t(\mu)$, define $\hat{f}$ on $bpe(P^t(\mu))$ by $\hat{f}(z) = \int f(\xi)K_z(\xi)d\mu(\xi)$. Observe that $\hat{f} = f a.e.\mu$ on $bpe(P^t(\mu))$ and in fact $z \mapsto \hat{f}(z)$ is analytic on $abpe(P^t(\mu))$. The set $abpe(P^t(\mu))$ support (μ) can be thought of as a set of overconvergence for $P^t(\mu)$. We now establish two results that address this topic of overconvergence.

Lemma 2.1. *Let μ be a finite, positive Borel measure with compact support in C. Let K be a compact subset of $abpe(P^t(\mu))$ and let $M = \sup\{|p(z)|^t : z \in K\}$, $p \in P$ and $||p||^t_{L^t(\mu)} = 1$. If $\mu(K) < 1/M$, then $||p||_{L^t(\mu)} \leq 1/(1 - M.\mu(K))^{1/t}.||P||_{L^t(\mu_{(C/K)})}$ for every polynomial p.*

Proof. By the definition of M , $|p(z)|^t \leq M.||p||^t_{L^t(\mu)}$ for every polynomial p. Therefore, $\int_C |p|^t d\mu = \int_{C/K} |P|^t d\mu + \int_K |p|^t d\mu \leq \int_{C/K} |p|^t d\mu + M.||p||^t_{L^t(\mu)}\mu(K)$, and hence $(1 - M.\mu(K)). \int_C |p|^t d\mu \leq \int_{C/K} |p|^t d\mu$. Consequently, $\int_C |p|^t d\mu \leq \frac{1}{(1-M.\mu(K))} \int_{C/K} |p|^t d\mu$ □

Proposition 2.2. *Let μ be a finite, positive Borel measure with compact support in C such that $D \subseteq abpe(P^t(\mu))$. If K is a compact subset of D, then $D \subseteq abpe(P^t(\mu_{(C/K)}))$*

Proof. Let $\rho = \max\{|z| : z \in K\}$. Now choose r such that $\rho \leq r \leq 1$. Since $\{z : |z| \leq r\}$ is a compact subset of $abpe(P^t(\mu))$, there exists $M_r > 1$ such that $|p(z)| \leq M_r.||p||_{L^t(\mu)}$ whenever $|z| \leq r$ and $p \in P$. Let $f_n(z) = (\frac{z}{r})^n$. Now, if $|z| = r$ and p is a polynomial, then $|p(z)| = |p(z).f_n(z)| \leq M_r||pf_n||_{L^t(\mu)} = M_r||p||_{L^t(\mu_n)}$ (where $d\mu_n = |f_n|^t d\mu$). Since $f_n \to o$ uniformly on K , there exists N such that $\mu_N(K).M_r \leq 1/2^t$. So, by Lemma 2.1, $|p(z)| \leq$

$2M_r||p||_{L^t(\mu_N(C/K))}$ whenever $p \in P$ and $|z| = r$. Now, there exists $c > 0$ such that $\mu_N \leq c.\mu$ and therefore $|p(z)| \leq 2cM_r.||p||_{L^t(\mu_N(C/K))}$ whenever $p \in P$ and $|z| = r$. By the Maximum Modulus Theorem, $\{z : |z| < r\} \subseteq abpe(P^t(\mu_{(C/K)})$. Since $\rho < r < 1$ is arbitrary, we conclude that $D \subseteq abpe(P^t(\mu_{(C/K)}))$ □

References

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