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Infinite 2D square network of identical capacitors with two missing bonds

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Abstract. The capacitance between any two arbitrary lattice sites of an infinite square network consisting of identical capacitors is studied analytically and numerically for a perturbed network. The perturbation is as a result of removing the bonds (i_0j_0) and (k_0l_0) from the perfect network. The equivalent capacitance is expressed in terms of the Lattice Green's Function (LGF) of the perturbed network. Solving Dyson's equation we express the LGF and the capacitance of the perturbed network in terms of those of the infinite perfect network. The asymptotic behavior of the perturbed capacitance is also studied. Finally, some numerical results are presented for the perturbed infinite square lattice, and a comparison is carried out for those of the perturbed and perfect infinite square network.

PACS. 02.70.Bf Finite-difference methods – 05.50.+q Lattice theory and statistics – 61.72.-y Defects and impurities in crystals; microstructure

1 Introduction

A classic and important problem in the electric circuit theory that has attracted the attention of many authors over many years is analyzing the electric circuit and the consideration of network resistances and impedances. Kirchhoff's [1] was the first one who formulated and studied the electric networks more than 150 years ago. The electric circuit theory is discussed in detail in a classic book by Van der Pol and Bremmer [2]. The basic problem in studying the electric circuit is the evaluation of the equivalent resistances and impedances which; in principle; can be carried out using traditional, but often tedious, analysis such as Kirchhoff's and Ohm's laws, there has been no fundamental solution.

In their book Van der Pol and Bremmer [2] derived the resistance between nearby points on the square lattice. Aitchson [3] gives an elegant and elementary solution for the problem of finding the resistance between two adjacent grid points of an infinite square lattice in which all the edges represent identical resistances R, and the result given in his work is $\frac{R}{2}$. Bartis [4] introduced how complex systems can be treated at the undergraduate level and he showed how to calculate the resistance between adjacent nodes of many lattices of 1-ohm resistors. During the 30 years that followed Bartis [4] work many authors [5–11] studied the resistance between adjacent sites of infinite lattices, where they used different methods (i.e. as an example; superposition of current distribution [5,6], and random walk theory [7]).

Later on, an important method [12–17] was used to calculate the effective resistance between any two arbitrary lattice points in many infinite lattices. The method is based on the LGF and it was introduced by Cserti [12]. The importance of this method lies in the facts that: (i) it can be used straightforwardly for complicated lattice structure (i.e. Body Centered Lattice-BCC-, and Face Centered Lattice-FCC-). (ii) The results presented by this method reflect the symmetry of the lattice structure. (iii) Some recurrence formulae for the resistance in infinite lattices can be derived using the LGF equations. (iv) Finally, the LGF method can be applied for both perfect infinite lattices, and for perturbed infinite lattices. Recently, Wu [18] obtained the resistance between two arbitrary nodes in a resistor network in terms of the eigenvalues and eigenfunctions of the Laplacian matrix associated with the network. Explicit formulae for two point resistances are deduced in his paper for regular lattices in one, two, and three-dimensions under various boundary conditions. Osterberg and Inan [19] extended Aitchison [3] and Bartis [4] work to the general problem of finding the effective resistance between two adjacent nodes of any d-dimensional infinite resistive lattice.

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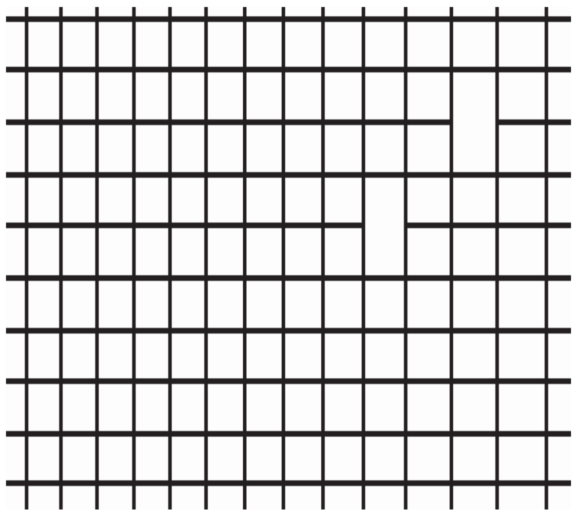

Fig. 1. Perturbation of an infinite square lattice consisting of identical capacitances C by removing the bond between the sites $\vec{r}_{i0}$, $\vec{r}_{j0}$ and the bond between the sites $\vec{r}_{k0}$, $\vec{r}_{l0}$. The capacitance $C(i,j)$ is calculated between arbitrary lattice sites $\vec{r}_i$ and $\vec{r}_j$.

The analysis of the capacitance of an infinite network of identical capacitors has been investigated recently [20–24]. The impedance of a standard ladder network of capacitors and inductors is studied by Van Enk [20], where the behavior of the impedance of a standard ladder network of capacitors and inductors is analyzed as a function of the size of the network. This behavior may be unstable in the absence of dissipation so that the limit of an infinite network is not well defined. Standard textbooks do not always treat this case correctly. In a previous work [21], we used the LGF method to calculate the capacitance between arbitrary lattice sites in a perfect infinite square lattice consisting of identical capacitors, some numerical results were given and the asymptotic behavior was also studied when the separation between the two sites goes to infinity. We have also [22] used the superposition of charge distribution in calculating the capacitance between two points in an infinite square grid of identical capacitors. Tzeng and Wu [23] introduced a formulation to determine the impedance between any two sites in an impedance network, where some numerical examples were given. In a recent work [24], we introduced the perturbed network of identical capacitors, where the LGF method is used to calculate the capacitance between arbitrary lattice sites in a perturbed infinite square lattice consisting of identical capacitors.

This work is organized as follows. In Section 2, we review the perfect case of an infinite square network of identical capacitors using Dirac's notation. In Section 3, the capacitance between arbitrary lattice sites $\vec{r}_i$ and $\vec{r}_j$ in the perturbed network is investigated. As an example (see Fig. 1), consider an infinite square lattice consisting of identical capacitances C. Removing two bonds from this perfect lattice results in a perturbed lattice. The problem is finding the equivalent capacitance $\frac{C(i,j)}{C}$. Finally, in Section 4, some numerical results are presented along different directions, and a comparison is carried out with the capacitances of the perfect network.

2 Perfect case

In this section, we review the formalism of the perfect infinite d-dimensional network using Dirac's notation.

Consider a perfect d-dimensional infinite lattice consisting of identical capacitors of capacitance C each, and take all the lattice points to be specified by the position vector $\vec{r}$ given in the form:

$$\vec{r} = l_1\vec{a}_1 + l_2\vec{a}_2 + \ldots + l_d\vec{a}_d \tag{2.1}$$

where $l_1, l_2, \ldots, l_d$ are integers (positive, negative or zero), and $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_d$ are independent primitive translation vectors.

Let the potential at the site $\vec{r}_i$ be $V(\vec{r}_i)$ and assuming a charge $+Q$ enters the site $\vec{r}_i$ and a charge $-Q$ exits the site $\vec{r}_j$, while the charges are zero at all other lattice sites. Thus, we may write:

$$Q_m = Q[\delta_{mi} - \delta_{mj}] \quad \text{for all } m. \tag{2.2}$$

Then, according to Ohm's and Kirchhoff's laws we may write:

$$\frac{Q(\vec{r}_i)}{C} = \sum_{\vec{n}} [V(\vec{r}_i) - V(\vec{r}_i + \vec{n})] \tag{2.3}$$

where $\vec{n}$ are the vectors from site $\vec{r}$ to its nearest neighbors ($\vec{n} = \pm a_i$, $i = 1, 2, \ldots, d$). One can form two state vectors, V and Q at the site $\vec{r}_i$ such that:

$$V = \sum_i |i\rangle\, V_i$$
$$Q = \sum_i |i\rangle\, Q_i \tag{2.4}$$

where $V_i = V(\vec{r}_i)$ and $Q_i = Q(\vec{r}_i)$.

Here we assumed that $|i\rangle$, associated with the site $\vec{r}_i$, forms a complete orthonormal set, i.e. $\langle i|k\rangle = \delta_{ik}$ and $\sum_i |i\rangle\langle i| = 1$. Using equations (2.4) and (2.3) one gets:

$$\sum_j (z\delta_{ij} - \varDelta_{ij})\,\langle j|\, V = \frac{\langle i|\, Q}{C} \tag{2.5}$$

z is the number of neighbors of each lattice site (e.g. $z = 2d$ for a d-dimensional hypercubic lattice) and $\varDelta_{kl}$ is defined as:

$$\varDelta_{kl} = \begin{cases} 1, & \vec{r}_k, \vec{r}_l \quad \text{are nearest neighbors} \\ \text{zero}, & \text{otherwise.} \end{cases} \tag{2.6}$$

The summation in equation (2.5) is taken over all lattice sites. Multiplying equation (2.5) by $|i\rangle$ and summing over i, one gets:

$$\sum_{i,j} |i\rangle\,(\varDelta_{ij} - z\delta_{ij})\,\langle j|\, V = \frac{-Q}{C}. \tag{2.7}$$

Or one may write:

$$L_0 V = \frac{-Q}{C} \tag{2.8}$$

where

$$L_0 = \sum_{i,j} |i\rangle \left(\Delta_{ij} - z\delta_{ij}\right) \langle j|$$

L_0 is the so-called lattice Laplacian.

Similarly to the definition used in Economou [25], the LGF for an infinite perfect lattice can be defined as:

$$L_0 G_0 = -1. \tag{2.9}$$

The solution of equation (2.9), which is a Poisson-like equation, in its simple form, can be given as:

$$V = -\frac{L_0^{-1} Q}{C} = \frac{G_0 Q}{C}. \tag{2.10}$$

Inserting equation (2.2) into equation (2.10), we obtained:

$$\begin{aligned} V_k = \langle k| V &= \frac{\langle k| G_0 Q}{C}, \\ &= \frac{1}{C} \sum_m \langle k| G_0 |m\rangle Q_m, \\ &= \frac{Q}{C}[G_0(k,i) - G_o(k,j)] \end{aligned} \tag{2.11}$$

where $G_0(l,m) = \langle l| G_0 |m\rangle$ is the matrix element of the operator G_0 in the basis $|l\rangle$.

Finally, the capacitance between the sites $\vec{r}_i$ and $\vec{r}_j$ in an infinite d-dimensional network is then given as:

$$\frac{1}{C_0(i,j)} = \frac{V_i - V_j}{Q} = \frac{2}{C}[G_0(i,i) - G_0(i,j)]. \tag{2.12}$$

The above formula can be rewritten as:

$$C_0(i,j) = \frac{C}{2[G_0(i,i) - G_0(i,j)]} \tag{2.13}$$

where we have make used of the symmetry of the LGF.

Note that the above formalism is valid for any infinite network consisting of identical capacitors.

Using the above result, the equivalent capacitance between the origin and the site (l,m) in an infinite square network can easily be calculated by expressing $G_0(l,m)$ in terms of $G_0(0,0)$ and its derivatives [26]. Or, by using the so-called recurrence formulae presented in reference [22]. The same results are obtained using the charge distribution method [23].

For comparison reasons some of the calculated values in reference [22,23] are presented in Table 1.

To study the asymptotic form of the capacitance for large separation between the two lattices sites $\vec{r}_i$ and $\vec{r}_j$ in an infinite square network one can write [24]:

$$\frac{C_0(i,j)}{C} = \frac{1}{\frac{1}{\pi}\left(\mathrm{Ln}\sqrt{(j_x - i_x)^2 + (j_y - i_y)^2} + \gamma + \frac{\mathrm{Ln}8}{2}\right)}. \tag{2.14}$$

As the separation between the sites $\vec{r}_i$ and $\vec{r}_j$ goes to infinity then one finds that

$$\frac{C_0(i,j)}{C} \to 0. \tag{2.15}$$

The above result can be explained as a parallel capacitance with infinite potential difference between its plates, which is obvious due to the large separation between the two sites which means that the potential difference goes to infinity ($C = Q/V$).

3 Perturbed case

In this section, the capacitance between the sites $\vec{r}_i$ and $\vec{r}_j$ in an infinite networks consisting of identical capacitors is derived when two bonds from the perfect infinite network are removed (i.e. the network is perturbed as in Fig. 1).

The charge contribution δQ_{i1} at the site $\vec{r}_i$ due to the bond $(i_0 j_0)$ is given by:

$$\begin{aligned} \frac{\delta Q_{i1}}{C} &= \delta_{ii_0}(V_{i_0} - V_{j_0}) + \delta_{ij_0}(V_{j_0} - V_{i_0}) \\ &= \langle i|i_0\rangle\left(\langle i_0| - \langle j_0|\right)V + \langle i|j_0\rangle\left(\langle j_0| - \langle i_0|\right)V \\ &= \langle i|\left(|i_0\rangle - |j_0\rangle\right)\left(\langle i_0| - \langle j_0|\right)V \\ \frac{\delta Q_{i1}}{C} &= \langle i| L_1 V \end{aligned} \tag{3.1}$$

where the operator L_1 has the form

$$L_1 = \left(|i_0\rangle - |j_0\rangle\right)\left(\langle i_0| - \langle j_0|\right) \tag{3.2}$$

and $\langle n|m\rangle = \delta_{nm}$ is used.

Replacing the bond $(i_0 j_0)$ by $(k_0 l_0)$ then, the charge contribution δQ_{i2} at the site $\vec{r}_i$ due to the bond $(k_0 l_0)$ can be given according to equation (3.1) as:

$$\frac{\delta Q_{i2}}{C} = \langle i| L_2 V \tag{3.3}$$

where the operator L_2 has the form

$$L_2 = \left(|k_0\rangle - |l_0\rangle\right)\left(\langle k_0| - \langle l_0|\right). \tag{3.4}$$

Now, removing the bonds $(i_0 j_0)$ and $(k_0 l_0)$ from the infinite perfect network then, one can write the charge Q_i at the site $\vec{r}_i$ as:

$$(-L_0 V)_i - \frac{\delta_{i1}}{C} - \frac{\delta_{i2}}{C} = \frac{Q_i}{C}. \tag{3.5}$$

Or, simply;

$$LV = -\frac{Q_i}{C} \tag{3.6}$$

where $L = L_{01} + L$, and $L_{01} = L_0 + L_1$.

Similarly to the case of an infinite perfect lattice, the LGF for the infinite perturbed lattice (i.e. the lattice formed after removing two bonds) can be written as:

$$LG = -1. \tag{3.7}$$

Table 1. Calculated values for the capacitance of an infinite square lattice between the origin and the site $j = (j_x, 0)$, for a perfect square lattice ($C_0(i,j)/C$); perturbed square lattice due to removing only the bond between $(0,0)$ and $(1,0)-(C_{01}(i,j)/C)-$; perturbed square lattice due to removing the bonds between $(0,0)$, $(1,0)$ and $(1,0)$, $(2,0)-(C_1(i,j)/C)-$; perturbed square lattice due to removing the bonds between $(0,0)$, $(1,0)$ and $(2,0)$, $(3,0)-(C_2(i,j)/C)-$; and finally, perturbed square lattice due to removing the bonds between $(1,0)$, $(2,0)$ and $(2,0)$, $(3,0)-(C_3(i,j)/C)$.

$j = (j_x, 0)$	$C_0(i,j)/C$	$C_{01}(i,j)/C$	$C_1(i,j)/C$	$C_2(i,j)/C$	$C_3(i,j)/C$
$(10,0)$	0.8014	0.7195	0.7163	0.7171	0.7195
$(9,0)$	0.8238	0.7362	0.7304	0.7324	0.7300
$(8,0)$	0.8502	0.7556	0.7459	0.7498	0.7527
$(7,0)$	0.8822	0.7785	0.7625	0.7693	0.7709
$(6,0)$	0.9223	0.8063	0.7797	0.7911	0.7887
$(5,0)$	0.9748	0.8408	0.7956	0.8129	0.8000
$(4,0)$	1.0482	0.8849	0.8029	0.8227	0.7817
$(3,0)$	1.1620	0.9421	0.7745	0.7502	0.6398
$(2,0)$	1.3759	1.0092	0.6124	0.9922	0.9537
$(1,0)$	2	1	0.8999	0.9651	0.9706
$(0,0)$	∞	∞	∞	∞	∞
$(-1,0)$	2	1.8610	1.7145	1.8273	1.8395
$(-2,0)$	1.3759	1.2597	1.2142	1.2496	1.2565
$(-3,0)$	1.1620	1.0601	1.0391	1.0554	1.0599
$(-4,0)$	1.0482	0.9563	0.9454	0.9537	0.9563
$(-5,0)$	0.9748	0.8902	0.8845	0.8888	0.8896
$(-6,0)$	0.9223	0.8433	0.8404	0.8425	0.8417
$(-7,0)$	0.8822	0.8076	0.8064	0.8072	0.8049
$(-8,0)$	0.8502	0.7793	0.7789	0.7792	0.7754
$(-9,0)$	0.8238	0.7561	0.7560	0.7561	0.7509
$(-10,0)$	0.8014	0.7365	0.7365	0.7366	0.7303

To find the equivalent capacitance between any lattice sites $\vec{r}_i$ and $\vec{r}_j$, we assume the charge distribution to be as in equation (2.2).

Now, the simplest solution of equation (3.6) is

$$V = -\frac{Q_i}{C} L^{-1}. \quad (3.8)$$

Using equation (3.7), then equation (3.8) becomes:

$$V = \frac{Q_i}{C} G. \quad (3.9)$$

To obtain the potentials at different sites, insert equation (2.10) into equation (3.9) one gets;

$$V_k = \frac{Q}{C}[G(k,i) - G(k,j)]. \quad (3.10)$$

Using equations (2.10) and (2.11), the capacitance between the two lattice sites $\vec{r}_i$ and $\vec{r}_j$ can be written as

$$\frac{1}{C(i,j)} = \frac{V_i - V_j}{Q}. \quad (3.11)$$

Inserting equation (3.10) into equation (3.11) we obtain:

$$\frac{C}{C(i,j)} = [G(i,i) - G(i,j) + G(j,j) - G(j,i)]. \quad (3.12)$$

Note here that $G(i,i) \neq G(j,j)$ since that the translational symmetry is broken due to the removed two bonds $(i_0 j_0)$ and $(k_0 l_0)$, but $G(i,j) = G(j,i)$.

According to Cserti et al. [13], the perturbed Green's function is given by:

$$G_{01}(i,j) = \langle i | G_{01} | j \rangle = G_0(i,j) + \frac{[G_0(i,i_0) - G_0(i,j_0)]\,[G_0(i_0,j) - G_0(j_0,j)]}{1 - 2\,[G_0(i_0,i_0) - G_0(i_0,j_0)]}. \quad (3.13)$$

The matrix elements of G for two removed bonds can be expressed in terms of the matrix elements of G_{01} by iteration procedure:

$$G(i,j) = G_{01}(i,j) + \frac{[G_{01}(i,k_0) - G_{01}(i,l_0)]\,[G_{01}(k_0,j) - G_{01}(l_0,j)]}{1 - [G_{01}(k_0,k_0) + G_{01}(l_0,l_0) - 2G_{01}(l_0,k_0)]}. \quad (3.14)$$

Finally, inserting equation (3.14) into equation (3.12) then, the equivalent capacitancebetween any two lattice sites $\vec{r}_i$ and $\vec{r}_j$ in the perturbed network (i.e. two bonds are removed) can be written as:

$$\frac{C}{C(i,j)} = G_{01}(i,i) + G_{01}(j,j) - 2G_{01}(i,j) + \frac{1}{1-[G_{01}(k_0,k_0)+G_{01}(l_0,l_0)-2G_{01}(k_0,l_0)]} \times \Big[[G_{01}(i,k_0)-G_{01}(i,l_0)]\,[G_{01}(k_0,i)-G_{01}(l_0,i)] + [G_{01}(j,k_0)-G_{01}(j,l_0)]\,[G_{01}(k_0,j)-G_{01}(l_0,j)] \times [G_{01}(i,k_0)-G_{01}(i,l_0)]\,[G_{01}(k_0,j)-G_{01}(l_0,j)] + [G_{01}(j,k_0)-G_{01}(j,l_0)]\,[G_{01}(k_0,i)-G_{01}(l_0,i)] \Big]. \tag{3.15}$$

Equation (3.15) consists of three parts, part I is

$$\text{Part I} = G_{01}(i,i) + G_{01}(j,j) - 2G_{01}(i,j),$$

and after some lengthily but straight-forward algebra it becomes:

$$\text{Part I} = \frac{C}{C_{01}(i,j)} = \frac{1}{C_0(i,j)} + \frac{\left[\frac{1}{C_0(i,j_0)} + \frac{1}{C_0(j,i_0)} - \frac{1}{C_0(i,i_0)} - \frac{1}{C_0(j,j_0)}\right]^2}{4\left[1-\frac{1}{C_0(i_0,j_0)}\right]}. \tag{3.16}$$

Equation (3.16) is exactly the same as that in reference [24], which is due to removing the bond (i_0j_0).

Taking $j_0 \to i_0$, equation (3.16) reduces to equation (2.13). The perturbed problem is reduced into the perfect one.

Part II is:

$$\text{Part II} = \frac{1}{1-[G_{01}(k_0,k_0)+G_{01}(l_0,l_0)-2G_{01}(k_0,l_0)]}. \tag{3.17}$$

Comparing part II with part I, we can write:

$$\frac{1}{1-[G_{01}(k_0,k_0)+G_{01}(l_0,l_0)-2G_{01}(k_0,l_0)]} = \frac{1}{1-\frac{1}{C'_{01}(k_0,l_0)}}. \tag{3.18}$$

Part III is:

$$\text{Part III} = \Big[[G_{01}(i,k_0)-G_{01}(i,l_0)]\,[G_{01}(k_0,i)-G_{01}(l_0,i)] + [G_{01}(j,k_0)-G_{01}(j,l_0)]\,[G_{01}(k_0,j)-G_{01}(l_0,j)] - [G_{01}(i,k_0)-G_{01}(i,l_0)]\,[G_{01}(k_0,j)-G_{01}(l_0,j)] - [G_{01}(j,k_0)-G_{01}(j,l_0)]\,[G_{01}(k_0,i)-G_{01}(l_0,i)] \Big] = \Big[[G_{01}(i,k_0)-G_{01}(i,l_0)] - [G_{01}(j,k_0)-G_{01}(j,l_0)] \Big]^2. \tag{3.19}$$

Insert equation (3.13) into equation (3.19), one gets after some straight forward but lengthy algebra:

$$\text{Part III} = \Big[\Big\{ G_0(i,k_0) + G_0(j,l_0) - G_0(i,l_0) - G_0(j,k_0) + \frac{1}{1-2[G_0(i_0,i_0)-G_0(i_0,j_0)]}(G_0(i,i_0)+G_0(j,j_0) - 2G_0(i,j_0))(G_0(j_0,l_0)+G_0(j_0,k_0) - G_0(i_0,l_0) - G_0(i_0,k_0))\Big\}^2\Big]. \tag{3.20}$$

Inserting equation (2.12) into the above equation and making use of the symmetry of the perfect LGF, one gets:

$$\text{Part III} = \left[\left\{ \frac{\frac{1}{C_0(j,k_0)} + \frac{1}{C_0(i,l_0)} - \frac{1}{C_0(j,l_0)} - \frac{1}{C_0(j,k_0)}}{2} + \frac{1}{1-\frac{1}{C_0(i_0,j_0)}} \times \left(\frac{\frac{2}{C_0(i,j_0)} - \frac{1}{C_0(i,i_0)} - \frac{1}{C_0(j,j_0)}}{2} \right) \times \left(\frac{\frac{1}{C_0(i_0,k_0)} + \frac{1}{C_0(i_0,l_0)} - \frac{1}{C_0(j_0,l_0)} - \frac{1}{C_0(j_0,k_0)}}{2} \right) \right\}^2 \right]. \tag{3.21}$$

$$\frac{C(i,j)}{C}=\frac{1}{\frac{1}{C_{01}(i,j)}+\frac{1}{1-\frac{1}{C'_{01}(k_0,l_0)}}\left\{C_{\mathrm{I}}(ij;k_0l_0)+\frac{1}{1-\frac{1}{C_0(i_0,j_0)}}C_{\mathrm{II}}(ij;i_0j_0)C_{\mathrm{III}}(ij;k_0l_0)\right\}^2}. \tag{3.22}$$

$$\frac{C'_{01}(k_0,l_0)}{C}=\frac{1}{\frac{1}{C_0(k_0,l_0)}+\frac{\left[\frac{1}{C_0(k_0,j_0)}+\frac{1}{C_0(l_0,i_0)}-\frac{1}{C_0(k_0,i_0)}-\frac{1}{C_0(l_0,j_0)}\right]^2}{4\left[1-\frac{1}{C_0(i_0,j_0)}\right]}} \tag{3.23}$$

Finally, from equations (3.16), (3.18) and (3.21), equation (3.15) becomes:

see equation (3.22) above

where $C'_{01}(k_0,l_0)$ is the capacitance between the two ends of the second removed bond (k_0,l_0) as affected from the first removed bond (i_0,j_0):

see equation (3.23) above

where $C_{\mathrm{I}}(ij;k_0l_0)$, $C_{\mathrm{II}}(ij;i_0j_0)$, and $C_{\mathrm{III}}(i_0j_0;k_0l_0)$ are defined as follows:

$$C_{\mathrm{I}}(ij;k_0l_0)=\frac{\frac{1}{C_0(j,k_0)}+\frac{1}{C_0(i,l_0)}-\frac{1}{C_0(j,l_0)}-\frac{1}{C_0(i,k_0)}}{2} \tag{3.24}$$

$$C_{\mathrm{II}}(ij;i_0j_0)=\frac{\frac{2}{C_0(i,j_0)}-\frac{1}{C_0(i,i_0)}-\frac{1}{C_0(j,j_0)}}{2} \tag{3.25}$$

$$C_{\mathrm{III}}(i_0j_0;k_0l_0)=\frac{\frac{1}{C_0(i_0,k_0)}+\frac{1}{C_0(i_0,l_0)}-\frac{1}{C_0(j_0,l_0)}-\frac{1}{C_0(j_0,k_0)}}{2}. \tag{3.26}$$

From equation (3.22), one can see that $C(i,j)$ is always less than $C_{01}(i,j)$, and from equation (3.16) $C_{01}(i,j)$ is always less than $C_0(i,j)$. This means that $C(i,j)$ is always less than $C_0(i,j)$.

To check our result. Taking $l_0 \to k_0$ and $j_0 \to i_0$, then one can see that equation (3.16) reduces to equation (2.13). This means that the perturbed problem is reduced to the perfect one.

Finally, to study the asymptotic behavior of the equivalent capacitance $C(i,j)$ as i or/and j goes to infinity, then from the definitions of $C_{\mathrm{I}}(ij;k_0l_0)$, $C_{\mathrm{II}}(ij;i_0j_0)$, $C_{\mathrm{III}}(i_0j_0;k_0l_0)$ and from equations (2.15) and (3.22) one finds that:

$$C(i,j)\to C_{01}(i,j)\to C_0(i,j)\to 0. \tag{3.27}$$

This means, that the effect of the removed bonds vanishes.

4 Numerical results and discussion

In this section, numerical results are presented for an infinite square lattice including both the perfect and the four perturbed cases (i.e. the bond between the sites $(0,0)$ and $(1,0)$ is removed alone, the two bonds between the sites $(0,0)$, $(1,0)$ and $(1,0)$, $(2,0)$ are removed, the two bonds between the sites $(0,0)$, $(1,0)$ and $(2,0)$, $(3,0)$ are removed and finally, the two bonds between the sites $(1,0)$, $(2,0)$ and $(2,0)$, $(3,0)$ are removed). The capacitance between the origin and the site $j=(j_x,0)$ in both an infinite perfect and perturbed square network is calculated.

For the infinite square network, we [21,22] calculated the capacitance between the origin and the lattice site $j=(j_x,0)$ (i.e. $(C_0(i,j)/C)$) using two different methods (i.e. LGF method and Charge distribution method); some of the calculated values are shown in Table 1 for comparison reasons.

On the perturbed (i.e. only the bond between the sites $(0,0)$ and $(1,0)$ is removed) square lattice the capacitance between the origin and the site $j=(j_x,0)$ (i.e. $(C_{01}(i,j)/C)$) has been calculated in reference [24] and some of the calculated values are shown in Table 1 for comparison reasons.

In this work, the site $\vec{r}_i$ is fixed (i.e. taken to be $(0,0)$) while the site $\vec{r}_j$ is moved along the line of the removed bonds (i.e. [10]-direction). Here we considered three cases; first, when the two removed bonds are between the sites $i_0=(0,0)$, $j_0=(1,0)$ and $k_0=(1,0)$, $l_0=(2,0)$, where, our calculated values of the capacitances (i.e. $(C_1(i,j)/C)$) are arranged in Table 1. In the second case, the two removed bonds are taken to be between $i_0=(0,0)$, $j_0=(1,0)$ and $k_0=(2,0)$, $l_0=(3,0)$, again our calculated values of the capacitances (i.e. $(C_2(i,j)/C)$) are arranged in Table 1. Finally, the two removed bonds are taken to be between $i_0=(1,0)$, $j_0=(2,0)$ and $k_0=(2,0)$, $l_0=(3,0)$, again our calculated values of the capacitances (i.e. $(C_3(i,j)/C)$) are arranged in Table 1.

In Figures 2–4 the capacitance for both the perfect and the above three perturbed cases are plotted as a function of j_x. It is clear from these figures that the perturbed capacitance is not symmetric along [10]-direction due to the fact that the inversion symmetry of the lattice is broken along this direction as a result of removing the two bonds, while the perfect capacitance is symmetric along

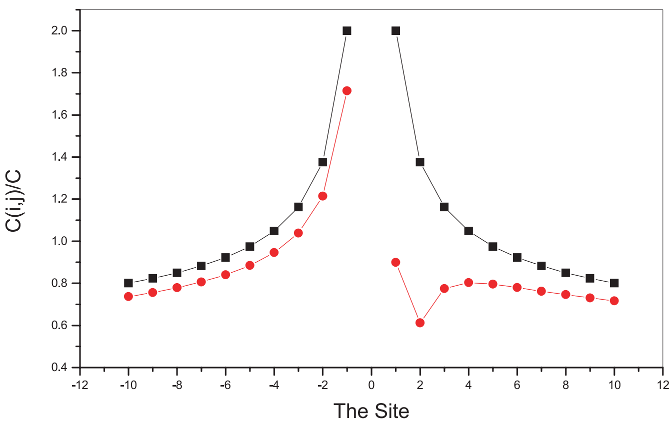

Fig. 2. The capacitance on the perfect (■) and the perturbed (●) square infinite lattice between $i = (0, 0)$ and $j = (j_x, 0)$ along the [10]-direction as a function of j_x. The removed bonds are between $i_0 = (0, 0)$, $j_0 = (1, 0)$ and $k_0 = (1, 0)$, $l_0 = (2, 0)$.

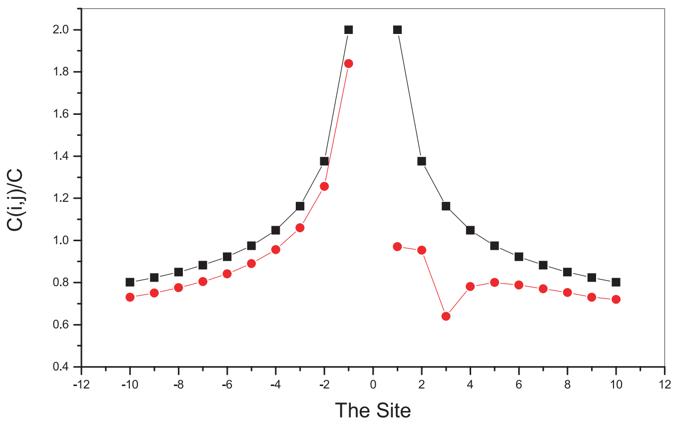

Fig. 4. The capacitance on the perfect (■) and the perturbed (●) square infinite lattice between $i = (0, 0)$ and $j = (j_x, 0)$ along the [10]-direction as a function of j_x. The removed bonds are between $i_0 = (1, 0)$, $j_0 = (2, 0)$ and $k_0 = (2, 0)$, $l_0 = (3, 0)$.

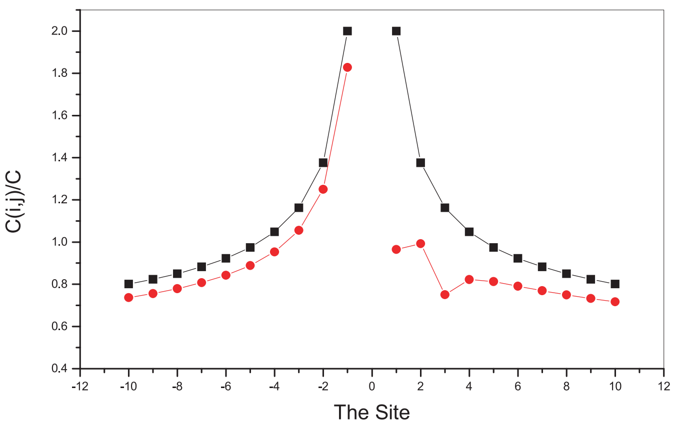

Fig. 3. The capacitance on the perfect (■) and the perturbed (●) square infinite lattice between $i = (0, 0)$ and $j = (j_x, 0)$ along the [10]-direction as a function of j_x. The removed bonds are between $i_0 = (0, 0)$, $j_0 = (1, 0)$ and $k_0 = (2, 0)$, $l_0 = (3, 0)$.

that direction since there is no broken bond along this direction. Also, one can see that the capacitance when two bonds are broken is always smaller than that when one bond is broken which in turn is always smaller than the perfect capacitance as mentioned before. In general, one can say that as the number of broken bonds increases in an infinite square lattice the perturbed capacitance decreases (i.e. approaches the perfect one, but never decreases than it). Remember that the capacitance between the two ends of the removed bond (i.e. in the case where one bond is removed) is equal to C, and this is explained in reference [21]. While the capacitance between the ends of the broken bond (i_0j_0) in the present of any other removed bond is never equal to C, this can be explained as in the first case there is an inversion symmetry around the broken bond, but in the second case the inversion symmetry around the broken bond (i_0j_0) is now broken due to the presence of the another broken bond.

From the calculated values and Figures 2–4, one can see that as the second removed bond (k_0l_0) is shifted away from the first broken bond (i_0j_0), then the calculated capacitance $C(i, j)$ increases and approaches those of the calculated values for $C_{01}(i, j)$ more rapidly and for large separation between the two sites $\vec{r}_i$ and $\vec{r}_j$ then $C(i, j) \to C_{01}(i, j) \to C_0(i, j)$ more rapidly (i.e. see calculated values for $C_2(i, j)$, $C_1(i, j)$, $C_{01}(i, j)$ and $C_0(i, j)$). Also, when the two broken bonds are shifted away together from the origin then the calculated capacitance $C(i, j)$ increases and approaches those of the calculated values for $C_{01}(i, j)$ more rapidly and for large separation between the two sites $\vec{r}_i$ and $\vec{r}_j$ then $C(i, j) \to C_{01}(i, j) \to C_0(i, j)$ more rapidly (i.e. see calculated values for $C_3(i, j)$, $C_1(i, j)$, $C_{01}(i, j)$ and $C_0(i, j)$).

Finally, as moving along the site $j = (-j_x, 0)$ we can see from Figures 2–4, that the calculated values $C_{01}(i, j)$, $C_1(i, j)$, $C_2(i, j)$ and $C_3(i, j)$ are very close specially when the separation between the two sites $\vec{r}_i$ and $\vec{r}_j$ increases. This is due to the fact that there are no broken bonds along this direction.

We feel that the content of this manuscript is helpful for electric circuit design and the method is instructive for electrical engineering. The method presented in this manuscript demonstrates a clear way for evaluating the equivalent capacitance of a network. Estimation of the edge contribution remains as a challenging problem, to be considered in future work.

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