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Generalizing the meaning of derivatives and integrals of any order differential equations by fuzzy-order derivatives and fuzzy-order integrals

Asia Khalaf Albzeirat [a], Israr Ahmad [b,*], Muhammad Zaini Ahmad [a], Shaher Momani [c], Fathilah Mohammad Alipiah [d]

[a] University Malaysia Perlis, Malaysia
[b] College of Applied Sciences, Nizwa, Oman
[c] The University of Jordan, Amman, Jordan
[d] University Utara, Malaysia

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ABSTRACT

This paper develops the correlation between fuzzy numbers and order of differential equations and overcomes the limitation in the existence of fractional order in the formulation of equation. In the view of fractional calculus, a new logic called fuzzy order by generalizing the meaning of derivatives and integrals of any order as fuzzy-order derivatives and fuzzy-order integral. We discuss D^{α}, where D^{α} is derivative of order α and α may be a triangular fuzzy number or trapezoidal fuzzy number, and propose to rewrite $D^{\alpha}y(x) = g(x, y(x))$, when $\alpha = (A, B, C)$ and A, B and $C \in N$ (where N is the set of natural numbers) and rewrite Riemann-Liouville integral, Riemann-Liouville derivative and Caputo fractional derivatives with respect to this new logic of fuzzy order. The proposed approach also covers multi cases, where the order is either integer or fractional. At the end, three numerical examples are presented to demonstrate the application of new logic, when the order of derivatives and integrals are given as triangular fuzzy numbers. These include time fractional heat equation represented as a time fuzzy-order heat equation and the time-fractional diffusion wave equation represented as a time-fuzzy-order diffusion wave equation.

1. Introduction

In this modern evolving world especially in the field of applied sciences and engineering, many continuous changes are observed and such changes require a thorough understanding of the nature and characteristics of the situation where undeniably certainty as well uncertainty prevail. Handling the issue of uncertainty in our daily lives is not just a matter of choosing a method or providing an answer to a question, but it is closely linked to the accuracy of the question itself and the consequences brought about by the question. These consequences could be the performance of a device or a machine, or it might be the efficiency of the daily instrument that surrounds us, be it natural or man-made. The consequences might be interrelated such as the evolution of global climate change and health consequence in pollution issues where there is a need of some form of appropriate questions to be raised so as to bring about mathematical formulas to explain them.

Questions are often raised as the results of suspicious knowledge or ignorance. An example of great importance was the incident of a falling apple which raised the question that inspired Newton to learn and reveal the concept of gravity and achieve a breakthrough through the discovery of the laws of gravity in 1966 (Ahmad, 2015; Riggs et al., 2015). The question regarding the falling apple has led to many other discoveries and concepts which in turn has played the role of eliminating doubts on further logical questions.

In 1695, an important question was raised by Leibniz in his letter to L'Hospital:

"*Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?*"

* Corresponding author.
E-mail addresses: israr_ahmad.niz@cas.edu.om (I. Ahmad), mzaini@unimap.edu.my (M.Z. Ahmad), s.momani@ju.edu.jo (S. Momani).

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L'Hospital was said to be somewhat curious about the question and replied by another simple question to Leibniz:

"*What if the order will be* $\frac{1}{2}$?"

Leibniz in a letter dated September 30, 1695, replied as follows:

"*It will lead to a paradox, from which one day useful consequences will be drawn*."

It is well known that this famous question by Leibniz has opened doors to numerous types of research in new areas in the field of differential equations (Selvam et al., 2015; Bernardis et al., 2016; Patil et al., 2015; Area et al., 2016; Yunquan and Chunfang, 2015). Although he himself did not really provide the answer to his own question, after 300 years, this particular question and the subsequent research work, which have led to a tremendous amount of applications in numerous areas such as science, engineering, astronomical science and in the interpretation of natural phenomena, have coined a new science which is known as the science of fractional differential equations (Li, 2015; Miller and Ross, 1993; Machado et al., 2014). The most popular definitions in relation to the question by Leibniz were formulated by Riemann, Liouville and Caputo where the definitions have been used in many theoretical and application studies.

Fractional differential equation can be defined as a differential equation which has non-integer order, and the following general formula can be considered:

$$D^{\alpha}y(t) = g(t, y(t)) \tag{1}$$

Eq. (1) was derived from the question by Leibniz in 1695, which was in the form of a statement directed to L'Hospital about the order of differential equations with questions that includes condition and possibility of having equations with a non-integer order raised through: "What if the order will be $\alpha = 0.5$?" (Agarwal et al., 2015; Leibniz, 1849; Carpinteri, 2014). Since the emergence of this question about the possibility of a fractional order, many successive studies on fractional derivatives have been carried out where such studies include the major ones were made in 1772 by Lagrange (West, 2014), in 1812 by Laplace (Zhou, 2014), Lacroix (Araci et al., 2015) in 1819 and in 1822 by Fourier (1822). In 1974, Diaz and Osler defined the fractional difference through a natural approach by allowing the index of differencing in a standard expression, either in real or complex numbers (Cafagna, 2007). It is apparently clear that a well-phrased question plays an important part in the evolution of a concept.

Fuzzy set theory was pioneered by Zadeh in 1965 (Zadeh, 1965). Zadeh stated that uncertainty and probability terms that have been earlier introduced in mathematics in theory as well as in applications are inadequate since the probability theory was limited (Zadeh, 1965). In light of this limitation and also the motivating question of how to explain uncertainty, Zadeh explored new concepts (Zadeh, 1966) with applicable rules of possibilities. These new concepts opened many paths to a new breakthrough in the field of scientific research, and thus providing answers to many ambiguous issues. He was questioning on how to mathematically define classes of objects where he himself has succeeded in providing the answer to the question. This new finding has been known as the fuzzy set theory. Goguen (Jalab and Ibrahim, 2015) began to relate and expanded Zadeh's findings to a bigger scope. Zadeh then continued to prove his previous work through a new study in the field of probability measures, while Chang (1968) made a study on fuzzy topological spaces. Among other applications, fuzzy set theory has also been used in decision-making (Chang and Zadeh, 1972). The fuzzy set theory became the language of redefining concepts through practical concepts and rules.

In relation of fuzzy set theory to the field of differential equations, the actual extension of fuzzy logic in differential equations was observed in the study of Chang and Zadeh in 1972 (Chang and Zadeh, 1972). In 1982, Dubois and Prade (1982) wrote about the fuzzy sets related to many applications such as differential equations and used the phrase 'fuzzy differential equation'. Furthermore, the authors (Dubois and Prade, 1982) introduced new applications in the fuzzy logic field specifically, in the integration of fuzzy mappings. Kaleva (1987) and Wang and Wu (1987) made applications of fuzzy in differential equations. Kaleva further worked on Cauchy problem for fuzzy differential equations (Kaleva, 1990). Kloeden (1991) wrote remarks on Peano-like theorems of fuzzy differential equations; Buckley and Qu (1991) solved the first-order fuzzy differential equations. The expansion of the fuzzy concept in numerical methods for solving differential equation has been studied (Friedman et al., 1999). Solution methods include Taylor methods (Abbasbandy and Viranloo, 2002), by Runge-Kutta method (Abbasbandy et al., 2004), predictor–corrector method (Allahviranloo et al., 2007), Nystrom method (Khastan and Ivaz, 2009), differential transformation method (Allahviranloo and Salahshour, 2011) and Laplace transforms (De Oliveira and Machado, 2013).

It is clear that the field of fuzzy differential equation has been expanding day by day. However, these studies were focused on the concept of fuzzy only within the scope of the initial values or the boundary values and such formulation may not represent the entire problem at hand. The current practice assumes the existence of certainty in the order of the equation. The concept of fuzzy was not taken into consideration in the order of the differential equation. Thus, the whole formulation is limiting and may not represent the entire problem at hand.

Therefore, we propose to raise a new question by restating the question posed by Leibniz in 1695 as follows:

"*Can the meaning of derivatives and integrals of any order be generalized to fuzzy-order derivatives and fuzzy-order integrals*?"

To be more specific, we propose to raise the following subsequent question which is parallel to L'Hospital reply to Leibniz:

"*What if the order will be fuzzy number*?"

We propose to view fractional calculus with a new logic which is a new order called fuzzy order. The proposition is to generalize the meaning of derivatives and integrals of any order as Fuzzy-order Derivatives and Fuzzy-order Integrals.

The rest of the paper is organized as follows. Section 2 includes some basic definitions in fractional calculus, followed by some main definitions in fuzzy set theory in Section 3. Section 4 presents equations with respect to the fuzzy order and provide new definition of differential calculus related to the new order. To employ the new logic, three numerical examples are given in Section 5. The paper concludes in Section 6.

2. Riemann-Liouville and Caputo definitions in fractional calculus

Fractional definitions, which have been originally formulated by Riemann and Caputo are given as follows.

Definition 1. (Salgado and Aguirre, 2016; Almeida, 2017; Momani and Odibat, 2007; Salahshour et al., 2012) (Riemann-Liouville integral)

The left and the right fractional integrals for Riemann-Liouville are defined as:

$$I^{\alpha}_{R,a}y(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\zeta)^{\alpha-1}y(\zeta)d\zeta \tag{2}$$

and

$$I^{\alpha}_{b,R}y(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\zeta - t)^{\alpha-1}y(\zeta)d\zeta \quad (3)$$

respectively, where $n-1<\alpha<n,$ and a,b are the terminal points of the interval $[a,b]$.

Definition 2. (Salgado and Aguirre, 2016; Almeida, 2017; Momani and Odibat, 2007; Salahshour et al., 2012) (Riemann-Liouville derivative)

The left and the right fractional derivatives for Riemann-Liouvilleare defined as:

$$D^{\alpha}_{R,a}y(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\zeta)^{n-\alpha-1}y(\zeta)d\zeta \quad (4)$$

and

$$D^{\alpha}_{R,t}y(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (\zeta-t)^{n-\alpha-1}y(\zeta)d\zeta \quad (5)$$

respectively, where $n-1\leqslant\alpha<n,$ and a,b are the terminal points of the interval $[a,b]$.

Definition 3. (Salgado and Aguirre, 2016; Almeida, 2017; Momani and Odibat, 2007; Salahshour et al., 2012; Zhang, 2014) (Caputo fractional derivatives)

The left and the right Caputo fractional derivatives are defined as:

$$D^{\alpha}_{c,a}y(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\zeta)^{n-\alpha-1}y^{(n)}(\zeta)d\zeta, \quad (6)$$

and

$$D^{\alpha}_{c,b}y(z) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (\zeta-t)^{n-\alpha-1}y^{(n)}(\zeta)d\zeta, \quad (7)$$

respectively, where $y^{(n)}\zeta = \frac{dy^n(\zeta)}{d\zeta^n}$ and $n-1\leqslant\alpha<n, n\in Z^+$.

3. Some important definitions in fuzzy theory

Definition 4. (Dubois and Prade, 1980). The Triangular fuzzy numbers ($TRFN$) are defined as follows:

$$TRFN(t) = \begin{cases} 0, & t<A \\ \frac{t-A}{B-A}, & A\leqslant t\leqslant B \\ \frac{C-t}{C-B}, & B\leqslant t\leqslant C \\ 0, & t>C \end{cases} \quad (8)$$

where $TRFN(t)\in R_F$, and its *r-cut* is defined as follows:

$$[TRFN(t)]^r = [A+r(B-A), C-r(C-B)], \quad \text{for} \quad r\in[0,1] \quad (9)$$

Definition 5. (Dubois and Prade, 1980). The Trapezoidal fuzzy numbers ($TLFN$) are defined as follows:

$$TLFN(t) = \begin{cases} 0, & t<A \\ \frac{t-A}{B-A}, & A\leqslant t\leqslant B \\ 1, & B\leqslant t\leqslant C \\ \frac{D-t}{D-C}, & C\leqslant t\leqslant D \\ 0, & t>D \end{cases} \quad (10)$$

where $TLFN(t)\in R_F$, and its *r-cut* is defined as follows:

$$[TLF(t)]^r = [A+r(B-A), D-r(D-C)], \quad \text{for} \quad r\in[0,1] \quad (11)$$

If $C=B$ $TLFN \equiv FRFN$.

4. The new concept of fuzzy order

In this section, we discuss differential equations with respect to fuzzy order and provide new definition of differential calculus related to the new order.

4.1. Fuzzy-order differential equation

By the logic of the question posed by Leibniz, rephrasing the Eq. (1) by redefining α as a fuzzy number as follows:

$$D^{\alpha}_{c}y(t) = g(t, y(t)),\ t>0,\ \alpha = TRFN. \quad (12)$$

By Definition 4 and Eq. (8), the new form of Eq. (12) yields:

$$\left[D^{(A,B,C)}_{c}y(t)\right]^r = g(t,y(t)) = \left[D^{A+r(B-A)}_{c}y(t), D^{C-r(C-B)}_{c}y(t)\right] = g(t,y(t)),\ t>0 \quad (13)$$

4.2. Fuzzy-order definitions

When we defined α as $TRFN$, then Eq. (2) is converted to the following form:

$$I^{TRFN}_{R,a}y(t) = \frac{1}{\Gamma(TRFN)}\int_a^t (t-\zeta)^{TRFN-1}y(\zeta)d\zeta. \quad (14)$$

where $n-1\leqslant TRFN<n,$ and a,b are the terminal points of the interval $[a,b]$.

Using the *r-cut* definition in Eqs. (8) and (9), Eq. (12) is converted to the following form:

$$\left[I^{TRF}_{R,a}y(t)\right]^r = \left[\frac{1}{\Gamma(TRF)}\int_a^t (t-\zeta)^{TRFN-1}y\zeta d\zeta\right]^r = \left[\frac{1}{\Gamma([TRFN]^r)}\int_a^t (t-\zeta)^{[TRFN]^r-1}y(\zeta)d\zeta\right] \quad (15)$$

By similar logic in Eqs. (12)–(15), we can rewrite the definitions in Eqs. (2)–(7) as follows:

Definition 6. The left Riemann-Liouville fuzzy-order integral $I^{TRFN}_{R,a}y(t)$ is given by:

$$\left[\frac{1}{\Gamma(A+r(B-A))}\int_a^t (t-\zeta)^{[A+r(B-A)]-1}y(\zeta)d\zeta, \frac{1}{\Gamma(C-r(C-B))}\int_a^t (t-\zeta)^{[C-r(C-B)]-1}y(\zeta)d\zeta\right], \quad (16)$$

where $A<A+r(B-A)<B, B<C-r(C-B)<C$and a is the left terminal point of the interval $[a,b]$.

Definition 7. The right Riemann-Liouville fuzzy-order integral $I^{TRFN}_{b,R}y(t)$ is given by:

$$\left[\frac{1}{\Gamma(A+r(B-A))}\int_t^b (\zeta-t)^{[A+r(B-A)]-1}y(\zeta)d\zeta, \frac{1}{\Gamma(C-r(C-B))}\int_t^b (\zeta-t)^{[C-r(C-B)]-1}y(\zeta)d\zeta\right], \quad (17)$$

where $A<A+r(B-A)<B, B<C-r(C-B)<C$and b is the right terminal point of the interval $[a,b]$.

The definition of the left and right Riemann-Liouville fuzzy-order derivatives are as follows.

Definition 8. The left Riemann-Liouville fuzzy-order derivative $D^{TRFN}_{R,a}y(t)$ is given by:

$$\begin{bmatrix} \frac{1}{\Gamma(B-(A+r(B-A)))}\frac{d^B}{dt^B}\int_a^t (t-\zeta)^{B-(A+r(B-A))-1}y(\zeta)d\zeta, \\ \frac{1}{\Gamma(C-(C-r(C-B)))}\frac{d^C}{dt^C}\int_a^t (t-\zeta)^{C-(C-r(C-B))-1}y(\zeta)d\zeta \end{bmatrix}, \tag{18}$$

where $A \leqslant A + r(B-A) < B, B \leqslant C - r(C-B) < C$and a is the left terminal point of the interval $[a, b]$.

Definition 9. The right Riemann-Liouville fuzzy-order derivative $D^{TRFN}_{b,R}y(t)$ can given by:

$$\begin{bmatrix} \frac{(-1)^B}{\Gamma(B-(A+r(B-A)))}\frac{d^B}{dt^B}\int_t^b (\zeta-t)^{B-(A+r(B-A))-1}y(\zeta)d\zeta, \\ \frac{(-1)^C}{\Gamma(C-(C-r(C-B)))}\frac{d^C}{dt^C}\int_t^b (\zeta-t)^{C-(C-r(C-B))-1}y(\zeta)d\zeta \end{bmatrix}, \tag{19}$$

where $A \leqslant A + r(B-A) < B, B \leqslant C - r(C-B) < C,$ and b is the right terminal point of the interval $[a, b]$.

The definition of left and right fuzzy-order derivatives in the Caputo sense are as follows.

Definition 10. The left Caputo fuzzy-order derivative is given by:

$$\begin{bmatrix} \frac{\langle ref10/\rangle}{\Gamma(B-(A+r(B-A)))}\int_a^t t-\zeta^{B-\alpha-1}y^{(B)}(\zeta)d\zeta, \\ \frac{\langle ref11/\rangle}{\Gamma(C-(C-r(C-B)))}\int_a^t (t-\zeta)^{C-\alpha-1}y^{(C)}(\zeta)d\zeta \end{bmatrix}, \tag{20}$$

where $y^{(B)}(\zeta) = \frac{dy^B(\zeta)}{d\zeta^B}, y^{(C)}(\zeta) = \frac{dy^C(\zeta)}{d\zeta C}, A \leqslant A + r(B-A) < B, B \leqslant C - r(C-B) < C.$

Definition 11. The right Caputo fuzzy-order derivative is given by:

$$\begin{bmatrix} \frac{(-1)^B}{\Gamma(B-(A+r(B-A)))}\int_t^b (\zeta-t)^{B-(A+r(B-A))-1}y^{(B)}(\zeta)d\zeta, \\ \frac{(-1)^C}{\Gamma(C-(C-r(C-B)))}\int_t^b (\zeta-t)^{C-(C-r(C-B))-1}y^{(C)}(\zeta)d\zeta \end{bmatrix}, \tag{21}$$

where $y^{(B)}(\zeta) = \frac{dy^B(\zeta)}{d\zeta^B}, y^C(\zeta) = \frac{dy^C(\zeta)}{d\zeta^C}, A \leqslant A + r(B-A) < B, B \leqslant C - r(C-B) < C.$

The previous definitions from (6)–(11) can be rewritten with the same procedure if α is given as *TLFN* in Eq. (12).

5. Numerical examples

In this section, three numerical examples are provided to validate the proposed logic and new definitions.

Example 1. The time fractional heat equation (Koch and Brady, 1988) is given by:

$$D^\alpha y(t) = a\Delta T, \tag{22}$$

where a is thermal diffusivity, T is the temperature and α is a positive real number. In the original problem, there is a wide range of choices for α and different values for the fractional order α give different heat conduction property, where in diffusion theory, it is called subdiffusion for $0 < \alpha < 1$ (Koch and Brady, 1988) and superdiffusion for $1 < \alpha < 2$ (Koch and Brady, 1988). The solutions in the original study are given for $\alpha = 0.6$ and 1.6, where this means the numerical solution for this equation requires two levels of order of derivatives with 0.6 located in first order fractional differential equation followed by 1.6 located in the second order fractional differential equation.

Using the new proposed logic, when α is defined as a *TRFN* where $\alpha = (0, 1, 2)$, the equation with this fuzzy order is able to cover all situations in diffusion theory. In this example, we have shown that the meaning of derivatives of any order can be generalized to fuzzy-order derivatives. Therefore, we can reformulate Eq. (22) by Eq. (8) as:

$$D^{\alpha=TRFN}y(t) = a\Delta T, \tag{23}$$

where a is thermal diffusivity, T is temperature and α is a fuzzy triangular number, based on Eq. (9) we can write the following formula for Eq. (23) as:

$$\begin{aligned} \left[D_c^{(A,B,C)}y(t)\right]^r &= g(t, y(t)) \\ &= \left[D^{\alpha_1}y(t), D^{\alpha_2}y(t)\right]\left[D_c^{A+r(B-A)}y(t), D_c^{C-r(C-B)}y(t)\right], \ t > 0 \end{aligned} \tag{24}$$

Clearly, Eq. (24) gives different heat conduction property when $\alpha = (A, B, C) = (0, 1, 2)$, where this new formulation gives different ranges for heat conduction property between subdiffusion and superdiffusion. The proposed fuzzy order is more suitable to describe the order of the fractional differential equation in this example. Using the new logic, we can observe the two diffusion situations for all $r \in [0, 1]$. This can be illustrated by the new formula of Eq. (22) as follows:

$$D^{\alpha=TRFN}y(t) = \begin{cases} Subdiffusion, & \alpha_1 = A + r(B-A), \ r \in [0,1] \\ Superdiffusion, & \alpha_2 = C - r(C-B), \ r \in [0,1]. \end{cases} \tag{25}$$

It is clear that the time fractional heat equation can be represented as a time fuzzy-order heat equation.

Example 2. For the equations in diffusion wave between subdiffusion and ballistic diffusion, such as time-fractional diffusion wave equation (Ott et al., 1990; Metzler and Klafter, 2000), where the time-fractional diffusion wave equation is obtained from the classical wave equation, which continues to the diffusion equation when $\alpha \to 1$ and to the wave equation when $\alpha \to 2$.

In typical diffusion α is 1, while other phenomena are called anomalous, where for $\alpha > 1, \alpha < 1$ and $\alpha = 2$, the phenomena are called superdiffusion, subdiffusion and ballistic diffusion, respectively.

By Definition 4, we can use the new logic by positioning the order given by fuzzy triangular numbers $\alpha = (0, 1, 2)$ that leads to cover cases of subdiffusion (weak diffusion) and normal diffusion for the first and second order differential equations, respectively as well as the cases of superdiffusion (strong diffusion) and ballistic diffusion for first level and second level for fractional differential equations, respectively. Hence, the time-fractional diffusion wave equation can be represented by the Time-fuzzy-order diffusion wave equation.

Example 3. By new logic presented in this paper, let us assumed the following equation:

$$D_c^\alpha y(t) = g(t, y(t_0)), y^{(\lfloor\alpha_1\rfloor)}(t_0) = k, y^{(\lfloor\alpha_2\rfloor)}(t_0) = q \ \text{ and } \ \alpha = (0, 1, 2), \tag{26}$$

where k and q are the initial conditions of problem and $\lfloor\alpha_i\rfloor$ denote the smallest and largest integer numbers nearest to α_i, i $= 1, 2$, respectively and () denote to the derivative of y.

From Eqs. (8) and (9), we can rewrite Eq. (26) as the follow:

$$D^{\alpha=TRFN}y(t) = \begin{cases} D_c^{\alpha_1}y(t), & \alpha_1 = r, r \in [0,1], \ y(t_0) = k, \\ D_c^{\alpha_2}y(t), & \alpha_2 = 2 - r, r \in [0,1], y(t_0) = k \text{ and } y^{(1)}(t_0) = q. \end{cases} \tag{27}$$

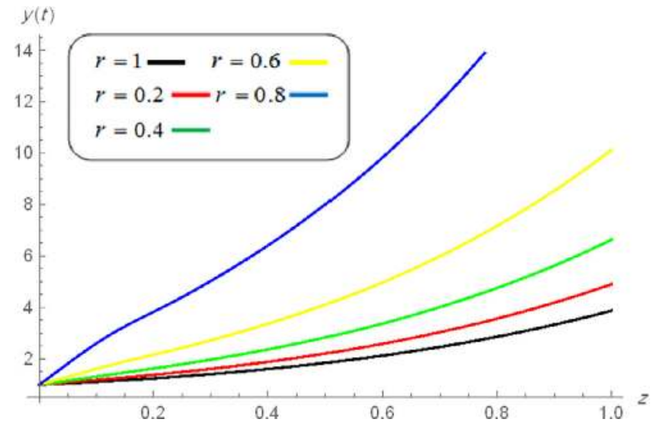

Fig. 1. Approximate solution of $y(t)$ when $\alpha_1 = r$ with different values of r.

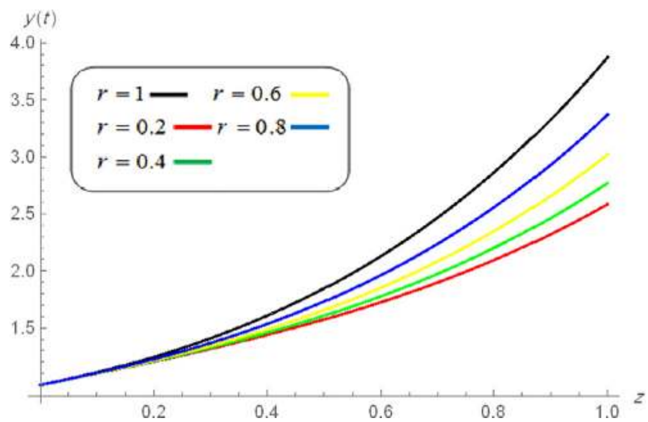

Fig. 2. Approximate solution of $y(t)$ when $\alpha_2 = 2 - r$ with different values of r.

Now, by using the methods in Albzeirat et al. (2017, 2018), we can solve Eq. (27) when $\alpha = r$ and $\alpha = 2 - r$. Figs. 1 and 2 illustrate the approximate solution of Eq. (26) when with different value of r.

6. Conclusion

This paper rephrased a renowned question by Liebnizin 1695 with respect to the proposed new logic called fuzzy-order and extended the fuzzy logic in order of differential equations. New fuzzy-order definitions have been introduced through the reformulation of the main definitions of fractional derivatives and integrals for Riemann-Liouville and Caputo. This work generalized the meaning of derivatives and integrals of any order to fuzzy-order derivatives and fuzzy-order integrals. The proposed new logic is employed for the generalization of time fractional heat equation and time fractional diffusion wave equation to time fuzzy-order heat equation and time-fuzzy-order diffusion wave equation, respectively as examples. It is noticeable that the actual application of this new logic presented here is still in early stages and could be investigated further in the future.

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