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DIFFERENTIAL TRANSFORM TECHNIQUE FOR SOLVING FIFTH-ORDER BOUNDARY VALUE PROBLEMS

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Abstract- In this paper we apply the differential transform method for solving fifth-order boundary value problems. The analytical and numerical results of the equations have been obtained in terms of convergent series with easily computable components. Three examples are given to illustrate the efficiency and implementation of the present method. Comparisons are made to confirm the reliability of the method. Differential transform technique may be considered as alternative and efficient for finding the approximate solutions of the boundary values problems.

Keywords- Fifth-order boundary value problems; Differential transform method; Numerical solution

1. INTRODUCTION

In this paper, we consider the general fifth-order boundary value problems of the type

$$y^{(5)}(x) = f(x, y), \quad a < x < b, \tag{1.1}$$

with boundary conditions

$$y(a) = A_1, \quad y'(a) = A_2, \quad y''(a) = A_3, \quad y(b) = B_1, \quad y'(b) = B_2, \tag{1.2}$$

where, $f = f(x, y)$ is a given continuous, linear or non-linear function of $y = y(x)$, $A_i (i = 1,2,3)$ and $B_i (i = 1,2)$ are real finite constants [10] .

Agarwal's book [19] contains theorems which detail the conditions for existence and uniqueness of solutions of such boundary value problems. This type of boundary value problems arises in the mathematical modeling of the viscoelastic flows and other branches of mathematical, physical and engineering sciences, see [2,4,6,8,19] and the references therein.

In [2,4], two numerical algorithms, namely, spectral Galerkin method and spectral collocation methods, were applied independently to address the numerical issues related to this type of problem. Moreover, a fifth-order boundary value problem was investigated by Khan [17] by using finite-difference methods. In [10], the sixth-degree B-spline functions were used and produced results which were improvements over other work. In [3], the Adomain decomposition method and a modified form of the Adomain decomposition method were used to investigate the fifth-order boundary value problem by Wazwaz. Also, the homotopy perturbation method has been used by Noor and Mohyud-Din [14] for solving fifth-order boundary value problems.

In this paper, we employ differential transform method [13] to solve Eq. (1.1) with boundary conditions (1.2). The concept of differential transform was first introduced by Zhou [13], in a study about electric circuit analysis. It is a semi-numerical-analytic-technique that formulizes Taylor series in a totally different manner. With this technique, the given differential equation and related boundary conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful to obtain exact and approximate solutions of linear and nonlinear differential equations. No need to linearization or discretization, large computational work and round-off errors are avoided. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations. The method is well addressed in [1,7,9,11,18,20].

2. DIFFERENTIAL TRANSFORM METHOD

As in [5,12,15,16], the basic definitions of differential transformation are introduced as follows:

Definition 2.1 If $f(t)$ is analytic in the time domain *T*, then let it will be differentiated continiously with respect to time *t*

$$\varphi(t_i,k)=\frac{d^k f(t)}{dt^k},\forall t\in T. \tag{2.1}$$

For $t=t_i$, then $\varphi(t,k)=\varphi(t_i,k)$, where *k* belongs to the set of non-negative integer, denoted as the *K*-domain. Therefore, Eq. (2.1) can be rewritten as

$$F(k)=\varphi(t_i,k)=\left[\frac{d^k f(t)}{dt^k}\right]\Bigg|_{t=t_i},\forall k\in K, \tag{2.2}$$

where $F(k)$ is called the spectrum of $f(t)$ at $t=t_i$ in the *K*-domain.

Definition 2.1 If $f(t)$ can be represented by Taylor's series, then it can be represented as

$$f(t)=\sum_{k=0}^{\infty}\left[(t-t_i)^k/k!\right]F(k). \tag{2.3}$$

Eq. (2.3) is called the inverse transform of $F(k)$. With the symbol *D* denoting differential transform process, and upon combining Eqs. (2.2) and (2.3), we obtain

$$f(t)=\sum_{k=0}^{\infty}\left[(t-t_i)^k/k!\right]F(k)\equiv D^{-1}\{F(k)\}. \tag{2.4}$$

Using the differential transform, a differential equation in the domain of interest can be transformed to an algebraic equation in the *K*-domain and then $f(t)$ can be obtained by finite-term Taylor series expansion plus a remainder, as

$$f(t)=\sum_{k=0}^{N}\left[(t-t_i)^k/k!\right]F(k)+R_{N+1}(t). \tag{2.5}$$

The fundamental mathematical operations performed by differential transform method are listed in Table 1.

Table 1.
The fundamental operations of differential transform method

Time function	Transformed function
$w(t)=\alpha u(t)\pm\beta v(t)$	$W(k)=\alpha U(k)\pm\beta V(k)$
$w(t)=\dfrac{d^m u(t)}{dt^m}$	$W(k)=\dfrac{(k+m)!}{k!}U(k+m)$
$w(t)=u(t)v(t)$	$W(k)=\sum_{l=0}^{k}U(l)V(k-l)$
$w(t)=t^m$	$W(k)=\delta(k-m)=\begin{cases}1, & if\ k=m,\\ 0, & if\ k\neq m.\end{cases}$
$w(t)=\exp(t)$	$W(k)=\dfrac{1}{k!}$
$w(t)=\sin(\omega t+\alpha)$	$W(k)=\dfrac{\omega^k}{k!}\sin\left(\dfrac{k\pi}{2}+\alpha\right)$
$w(t)=\cos(\omega t+\alpha)$	$W(k)=\dfrac{\omega^k}{k!}\cos\left(\dfrac{k\pi}{2}+\alpha\right)$

3. NUMERICAL RESULTS

Example 1 Consider the following linear boundary value problem

$$y^{(5)}(x)=y(x)-15e^x-10xe^x,\ \ 0<x<1, \tag{3.1}$$

with boundary conditions

$$y(0)=0,\ y'(0)=1,\ y''(0)=0,\ y(1)=0,\ y'(1)=-e. \tag{3.2}$$

The exact solution of the problem is:

$$y(x)=x(1-x)e^x. \tag{3.3}$$

Taking the differential transform of both sides of Eq. (3.1), we obtain

$$Y(k+5)=\frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)}\left(Y(k)-\frac{15}{k!}-10\sum_{l=0}^{k}\frac{\delta(l-1)}{(k-l)!}\right). \tag{3.4}$$

By using Eqs.(2.2) and (3.2), the following transformed boundary conditions at $x_0=0$ can be obtained:

$$Y(0)=0,\ Y(1)=1,\ Y(2)=0,\ \sum_{k=0}^{n} Y(k)=0,\ \sum_{k=0}^{n} kY(k)=-e. \tag{3.5}$$

Utilizing the recurrence relation in Eq. (3.4) and the transformed boundary conditions in Eq.(3.5), the following series solution up to $O(x^{13})$ is obtained:

$$\begin{aligned} y(x) &= x + ax^3 + bx^4 - \frac{x^5}{8} - \frac{x^6}{30} - \frac{x^7}{144} + \left(\frac{a}{6720} - \frac{1}{896}\right)x^8 + \left(\frac{b}{15120} - \frac{11}{72576}\right)x^9 \\ &\quad - \frac{x^{10}}{45360} - \frac{x^{11}}{403200} - \frac{x^{12}}{3991680} + O(x^{13}), \end{aligned} \tag{3.6}$$

where, according to Eq.(2.2),

$$a = \frac{y^{(3)}(0)}{3!} = Y(3),\ b = \frac{y^{(4)}(0)}{4!} = Y(4). \tag{3.7}$$

By taking $n = 12$, the following system of equations can be obtained from Eq. (3.5):

$$\begin{aligned} &\frac{6721a}{6720} + \frac{15121b}{15120} = -\frac{33267851}{39916800}, \\ &\frac{2521a}{840} + \frac{6721b}{1680} = -\left(e + \frac{4624181}{39916800}\right). \end{aligned} \tag{3.8}$$

From Eq. (3.8), a and b are evaluated numerically as

$$a = \text{-}0.499999772,\ \ b = \text{-}0.333333586. \tag{3.9}$$

Then Eq. (3.6) becomes

$$\begin{aligned} y(x) &= x - 0.499999772x^3 - 0.333333586x^4 - 0.125x^5 - 0.033333333x^6 \\ &\quad - 0.006944444x^7 - 0.001190476x^8 - 0.000173611x^9 - 0.000022046x^{10} \\ &\quad - 0.000002480x^{11} - 0.000000251x^{12} + O(x^{13}). \end{aligned} \tag{3.10}$$

Table 2 exhibits a comparison between the errors obtained by using the differential transform method (DTM) and the sixth-degree B-spline method in [10]. From the table, it is clear that the present method is more efficient and easy to implement as compared with B-spline technique.

Table 2
Error estimates

x	Analytical solution	Errors(DTM)	Errors(B-splines)
0.0	0.000000000	0.000000000	0.0000
0.1	0.099465383	0.000000000	-8.0E-3
0.2	0.195424441	0.000000000	-12.0E-3
0.3	0.283470350	0.000000000	-5.0E-3
0.4	0.358037927	0.000000000	3.0E-3
0.5	0.412180318	0.000000000	8.0E-3
0.6	0.437308512	0.000000000	6.0E-3
0.7	0.422888069	0.000000000	-0.000
0.8	0.356086549	0.000000000	9.0E-3
0.9	0.221364280	0.000000000	-9.0E-3
1	0.000000000	0.000000000	0.0000

Example 2 Consider the following nonlinear fifth-order boundary value problem

$$y^{(5)}(x) = e^{-x} y^2(x), \quad 0 < x < 1, \tag{3.11}$$

with boundary conditions

$$y(0) = 1, \; y'(0) = 1, \; y''(0) = 1, \; y(1) = y'(1) = e. \tag{3.12}$$

The theoretical solution for this problem is:

$$y(x) = e^x. \tag{3.13}$$

Taking the differential transform of (3.11), we obtain the following recurrence relation:

$$Y(k+5) = \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)} \left(\sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} \frac{(-1)^{(k-k_2)}}{(k-k_2)!} Y(k_1) Y(k_2 - k_1) \right) \tag{3.14}$$

The boundary conditions in Eq.(3.12) can be transformed at $x_0 = 0$ as follows:

$$Y(0) = 1, \; Y(1) = 1, \; Y(2) = 1/2, \; \sum_{k=0}^{n} Y(k) = e, \; \sum_{k=0}^{n} kY(k) = e. \tag{3.15}$$

Utilizing the recurrence relation in Eq. (3.14) and the transformed boundary conditions in Eq.(3.15), the following series solution up to $O(x^9)$ is obtained:

$$y(x) = 1 + x + \frac{x^2}{2} + ax^3 + bx^4 + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \left(\frac{a}{3360} - \frac{1}{40320} \right) x^8 + O(x^9). \tag{3.16}$$

By taking $n = 8$ and using Eqs. (3.14) and (3.15), we can obtain the following system of equations:

$$\begin{aligned} &\frac{3361a}{3360} + b = e - \frac{4819}{1920}, \\ &\frac{1261a}{420} + 4b = e - \frac{1723}{840}. \end{aligned} \tag{3.17}$$

We get from the above system:

$$a = 0.166651022, \quad b = 0.041685374. \tag{3.18}$$

Substituting (3.18) into (3.16) yields the series solution

$$\begin{aligned} y(x) &= 1 + x + 0.5x^2 + 0.166651022x^3 + 0.041685374x^4 + 0.008333333\text{x}^5 \\ &\quad + 0.001388889x^6 + 0.000198413x^7 + 0.000024797x^8 + O(x^9). \end{aligned} \tag{3.19}$$

Table 3 shows that the exact values and errors obtained by using the differential transform method and the sixth degree B-spline method [10]. From the numerical results in Table 3, it is clear that the differential transform method is more efficient and easy to implement as compared with B-spline technique.

Table 3
Error estimates

x	Analytical solution	Errors(DTM)	Errors (B-splines)
0.0	1.000000000	1.000000000	0.0000
0.1	1.105170918	1.4E-8	7.0E-4
0.2	1.221402758	9.5E-8	-7.2E-4
0.3	1.349858808	2.71E-7	4.1E-4
0.4	1.491824698	5.23E-7	4.6E-4
0.5	1.648721271	7.92E-7	4.7E-4
0.6	1.822118800	9.84E-7	4.8E-4
0.7	2.013752707	9.94E-7	3.9E-4
0.8	2.225540928	7.49E-7	3.1E-4
0.9	2.459603111	3.05E-7	1.6E-4
1	2.718281828	0.0000	0.0000

Example 3 For $x \in [-1,1]$, let us consider the following boundary value problem

$$y^{(5)}(x) = 19x\cos(x) + 2x^3\cos(x) + 41\sin(x) - 2x^2\sin(x) - xy(x), \tag{3.20}$$

with the boundary conditions

$$\begin{aligned} &y(-1) = y(1) = \cos(1), \\ &y'(-1) = -y'(1) = -4\cos(1) + \sin(1), \\ &y''(-1) = 3\cos(1) - 8\sin(1). \end{aligned} \tag{3.21}$$

The analytical solution of the above problem is:

$$y(x) = (2x^2 - 1)\cos(x). \tag{3.22}$$

By appliying the fundamental mathematical operations performed by differential transform, the differential transform of Eq.(3.20) is obtained as

$$Y(k+5) = \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)} \times \left(\begin{array}{l} 19\sum_{k_1=0}^{k}\delta(k_1-1)c(k-k_1) + 41s(k) + 2\sum_{k_1=0}^{k}\delta(k_1-3)c(k-k_1) - 2\sum_{k_1=0}^{k}\delta(k_1-2)s(k-k_1) \\ -\sum_{l=0}^{k}\delta(l-1)Y(k-l) \end{array} \right) \tag{3.23}$$

where $s(k)$ and $c(k)$ correspond to the differential transformation of $\sin(x)$ and $\cos(x)$ at $x_0 = 0$, respectively, which can be easily obtained from the definition of differential transform in Eq. (2.2) as follows:

$$S(k) = \begin{cases} \dfrac{(-1)^{\frac{(k-1)}{2}}}{k!}, & \text{if } k = \text{odd} \\ 0, & \text{if } k = \text{even} \end{cases} \qquad C(k) = \begin{cases} \dfrac{(-1)^{\frac{k}{2}}}{k!}, & \text{if } k = \text{even} \\ 0, & \text{if } k = \text{odd} \end{cases} \tag{3.24}$$

The boundary conditions in Eq. (3.21) can be transformed at $x_0 = 0$ as

$$\sum_{k=0}^{n} Y(k)(-1)^{k} = \cos(1), \quad \sum_{k=0}^{n} Y(k) = \cos(1),$$
$$\sum_{k=0}^{n} kY(k)(-1)^{k-1} = -4\cos(1)+\sin(1),$$
$$\sum_{k=0}^{n} kY(k) = 4\cos(1)-\sin(1), \tag{3.25}$$
$$\sum_{k=0}^{n} k(k-1)Y(k)(-1)^{k-2} = 3\cos(1)-8\sin(1),$$

where, n is a sufficiently large integer. By using the inverse transformation rule in Eq.(2.3), for $n=6$, we get

$$y(x) = a_0 + xa_1 + a_2x^2 + a_3x^3 + a_4x^4 + \left(\frac{1}{12} - \frac{a_0}{720}\right)x^6 + O(x^7). \tag{3.26}$$

where,

$a_0 = y(0) = Y(0),\ a_1 = y'(0) = Y(1),\ a_2 = y''(0)/2! = Y(2)$,

$a_3 = y'''(0)/3! = Y(3)$ and $a_4 = y^{(4)}(0)/4! = Y(4)$.

Also, by taking $n=6$, the following system of equations can be obtained from Eq.(3.25):

$$\frac{719a_0}{720} - a_1 + a_2 - a_3 + a_4 = \cos(1) - \frac{1}{12},$$
$$\frac{719a_0}{720} + a_1 + a_2 + a_3 + a_4 = \cos(1) - \frac{1}{12},$$
$$\frac{a_0}{120} + a_1 - 2a_2 + 3a_3 - 4a_4 = -4\cos(1) + \sin(1) + \frac{1}{2}, \tag{3.27}$$
$$-\frac{a_0}{120} + a_1 + 2a_2 + 3a_3 + 4a_4 = 4\cos(1) - \sin(1) - \frac{1}{2},$$
$$-\frac{a_0}{24} + 2a_2 - 6a_3 + 12a_4 = 3\cos(1) - 8\sin(1) - \frac{5}{2},$$

We get from the equation system (3.27):

$$a_0 = -1.008125221, a_1 = 0, a_2 = 2.521719441, a_3 = 0, a_4 = -1.058025422. \tag{3.28}$$

Then, by using the inverse transformation rule in Eq. (2.3), we get the following series solution

$$y(x) = -1.008125221 + 2.521719441x^2 - 1.058025422x^4 + 0.084733507x^6 - O(x^8).(3.29) \tag{3.29}$$

By continuing the same procedure for $n=12$, we get the following series solution:

$$y(x) = -0.999999938 + 2.499999854x^2 - 1.041666579x^4 + 0.084722222x^6 - 0.002802579x^8 + 0.000049879x^{10} - 0.000000553x^{12} + O(x^{14}). \quad (3.30)$$

Numerical results for $n = 6$ and $n = 12$ with comparison to the exact solution (3.22) are given in Table 4.

Table 4
Numerical results compared to the exact solution for Example 3

x	$n = 6$	$n = 12$	Exact
-1.0	0.540302306	0.540302306	0.540302306
-0.8	0.194620389	0.195077883	0.195077879
-0.6	-0.233472990	-0.231093951	-0.231093972
-0.4	-0.631388492	-0.626321435	-0.626321476
-0.2	-0.908943861	-0.901661195	-0.901661252
0.0	-1.000000000	-1.000000000	-1.000000000
0.2	-0.908943861	-0.901661195	-0.901661252
0.4	-0.631388492	-0.626321435	-0.626321476
0.6	-0.233472990	-0.231093951	-0.231093972
0.8	0.194620389	0.195077883	0.195077879
1.0	0.540302306	0.540302306	0.540302306

As the number of terms involved increase, one can observe that the series solution obtained by differential transform method converges to the series expansion of the exact solution (3.22).

4. CONCLUSION

In this paper, we have shown that the differential transform method can be used successfully for finding the solution of linear and nonlinear boundary value problems of fifth-order. It may be concluded that this technique is very powerful and efficient in finding semi analytical and numerical solutions for these types of boundary value problems.

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