



ORIGINAL ARTICLE

Differential transform method for solving singularly perturbed Volterra integral equations

Nurettin Doğan^a, Vedat Suat Ertürk^b, Shaher Momani^c, Ömer Akın^d, Ahmet Yıldırım^{e,*}

^a Gazi University, Technical Education Faculty Electronics and Computer Education Department, 06500 Teknikokullar, Ankara, Turkey

^b Department of Mathematics, Faculty of Arts and Sciences, Ondokuz Mayıs University, 55139 Samsun, Turkey

^c Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan

^d Department of Mathematics, Faculty of Arts and Sciences, TOBB Economics and Technology University, Söğütözü, 06530 Ankara, Turkey

^e Ege University, Science Faculty, Department of Mathematics, 35100 Bornova-İzmir, Turkey

Received 6 July 2010; accepted 9 July 2010

Available online 15 July 2010

KEYWORDS

Differential transform method;
Singularly perturbed problems;
Volterra integral equations

Abstract In this work, the applications of differential transform method were extended to singularly perturbed Volterra integral equations. To show the efficiency of the method, some singularly perturbed Volterra integral equations are solved as numerical examples. Numerical results show that the differential transform method is very effective and convenient for solving a large number of singularly perturbed problems with high accuracy.

© 2010 King Saud University. Production and hosting by Elsevier B.V. All rights reserved.

* Corresponding author.

E-mail addresses: ndogan@gazi.edu.tr (N. Doğan), vserturk@samsun.edu.tr (V.S. Ertürk), shaherm@yaho.com (S. Momani), omerakin@etu.edu.tr (Ö. Akın), ahmet.yildirim@ege.edu.tr (A. Yıldırım), ahmet.yildirim@ege.edu.tr (A. Yıldırım).

1018-3647 © 2010 King Saud University. Production and hosting by Elsevier B.V. All rights reserved.

Peer review under responsibility of King Saud University.

doi:10.1016/j.jksus.2010.07.013



Production and hosting by Elsevier

1. Introduction

In this paper we consider the singularly perturbed Volterra integral equations (Alnasr et al., 1997, 2000; Lange and Smith, 1988; Angell and Olmstead, 1987)

$$\varepsilon y(x) = g(x) + \int_0^x K(x, t, y(t)), \quad 0 \leq t \leq X, \quad (1.1)$$

where, ε is a small parameter satisfying $0 < \varepsilon < 1$ and where g and K are given smooth functions on $[0, X]$. Under appropriate conditions g and K , for every $\varepsilon > 0$, Eq. (1.1) has unique continuous solutions on $[0, T]$ (see, e.g., Alnasr et al., 1997; Brunner and Van Der Houwen, 1986). The singularly perturbed nature of (1.1) arises when the properties of the solution with $\varepsilon > 0$ are incompatible with those when $\varepsilon = 0$. For $\varepsilon > 0$, (1.1) is an integral equation of the second kind which typically is

well posed whenever K is sufficiently well behaved. When $\varepsilon = 0$, (1.1) is reduced to an integral equation of the first kind whose solution may well be incompatible with the case for $\varepsilon > 0$. The interest here is in those problems which do imply such an incompatibility in the behavior of y near $x = 0$. This suggests the existence of boundary layer near the origin where the solution undergoes a rapid transition (Alnasr et al., 1997, 2000; Lange and Smith, 1988; Angell and Olmstead, 1987).

The aim of our study is to employ the differential transform method (in short, DTM) Pukhov et al., 1980; Zhou et al., 1986 as an alternative to existing methods in solving the singularly perturbed Volterra integral problems and the method is implemented to four numerical examples. The concept of differential transform method was first introduced by Pukhov et al. (1980) and Zhou et al. (1986), who solved linear and nonlinear initial value problems in electric circuit analysis. It is a semi-numerical and semi-analytic technique that formulizes Taylor series in a totally different manner. With this technique, the given differential equation and its related boundary conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful to obtain the exact and approximate solutions of linear and nonlinear differential equations. No need to linearization or discretization, large computational work and round-off errors are avoided. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations. The method is well addressed in Ertürk and Momani (2007), Arıkoğlu and Özkol (2005), Ayaz (2004), Liu and Song (2007), Hassan (2008), Bildik et al. (2006) and Ertürk and Momani (2007).

The paper has been organized as follows. In Section 2, a brief description of the method is presented, while, in Section 3, four numerical examples are solved to demonstrate the applicability of the present method. The discussion on our results is given in Section 4.

2. Differential transform method

Let $f(r)$ be analytic in a domain R and let $r = r_0$ represent any point in R . Then, the function $f(r)$ is represented by a power series whose center is located at r_0 . The differential transform of the k th derivative of the function $f(r)$ in one variable is defined as follows:

$$F(k) = \frac{1}{k!} \left[\frac{d^k f(r)}{dr^k} \right]_{r=r_0}, \tag{2.1}$$

where, $f(r)$ is the original function and $(F(k))$ is the transformed function. The inverse transformation of the function $f(r)$ is defined by

$$f(r) = \sum_{k=0}^{\infty} F(k)(r - r_0)^k. \tag{2.2}$$

Combining Eqs. (2.1) and (2.2), one may write:

$$f(r) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k f(r)}{dr^k} \right]_{r=r_0} (r - r_0)^k. \tag{2.3}$$

Eq. (2.3) implies that the concept of differential transform method is derived from Taylor series expansion. However, the method does not evaluate the derivatives symbolically. An iterative procedure which is described by the transformed

Table 1 Operations of differential transform.

Original function	Transformed function
$f(x) = u(x) \pm v(x)$	$F(k) = U(k) + G(k)$
$f(x) = \alpha u(x)$	$F(k) = \alpha U(k)$
$f(x) = u(x)v(x)$	$F(k) = U(k)G(k)$
$f(x) = \frac{du(x)}{dx}$	$F(k) = (k + 1)U(k + 1)$
$f(x) = \frac{d^m u(x)}{dx^m}$	$F(k) = (k + 1)(k + 2) \dots (k + m)U(k + m)$
$f(x) = x^m$	$F(k) = \delta(k - m)$
$f(x) = e^{\lambda x}$	$F(k) = \frac{\lambda^k}{k!}$
$f(x) = \sin(\omega x + \alpha)$	$F(k) = \frac{\omega^k}{k!} \sin(\pi k/2 + \alpha)$
$f(x) = \cos(\omega x + \alpha)$	$F(k) = \frac{\omega^k}{k!} \cos(\pi k/2 + \alpha)$
$f(x) = \int_{x_0}^x u(t) dt$	$F(k) = \frac{U(k-1)}{k}, k \geq 1, F(0) = 0$
$f(x) = \int_{x_0}^x u(t)v(t) dt$	$F(k) = \frac{1}{k} \sum_{k_1=0}^{k-1} U(k_1)V(k - k_1 - 1), k \geq 1$
$f(x) = u(x) \int_{x_0}^x v(t) dt$	$F(k) = \sum_{k_1=1}^k \frac{1}{k_1} U(k - k_1)V(k_1 - 1), k \geq 1$

equations of the original functions can be used to calculate the related derivatives. In this study, we use the lower case letter to represent the original function and upper case letter to stand for the transformed function.

In actual applications, the function $f(r)$ is expressed by a finite series and Eq. (2.2) can be written as

$$f(r) = \sum_{k=0}^N F(k)(r - r_0)^k, \tag{2.4}$$

which implies that $\sum_{k=N+1}^{\infty} F(k)(r - r_0)^k$ is negligibly small. In this study, the convergence of the natural frequencies determines the value of N . The theorems which are frequently used in the transformation analysis are shown in Table 1.

3. Applications and numerical results

In order to illustrate the advantages and the accuracy of differential transform method for solving singularly perturbed Volterra integral equations, we have applied the method to four problems.

Example 1. We consider the following linear singularly perturbed Volterra integral equation discussed in Alnasr (2000) and Alnasr and Momani (2008):

$$\varepsilon y(x) = \int_0^x [1 + t - y(t)] dt. \tag{3.1}$$

The exact solution of this problem is known as

$$y(x) = x + 1 - \exp\left(\frac{-x}{\varepsilon}\right) - \varepsilon \left(1 - \exp\left(\frac{-x}{\varepsilon}\right)\right) \tag{3.2}$$

According to the operations of differential transformation given in Table 1, we have the following recurrence relation:

$$Y(k) = \frac{1}{\varepsilon} \left(\frac{\delta(k - 1) + \delta(k - 2) - Y(k - 1)}{k} \right) k \geq 1, Y(0) = 0. \tag{3.3}$$

Utilizing the recurrence relation in Eq. (3.3), we find

$$Y(1) = \frac{1}{\varepsilon},$$

$$Y(2) = -\frac{1}{2\varepsilon^2} + \frac{1}{2\varepsilon},$$

$$\begin{aligned}
 Y(3) &= \frac{1}{6\epsilon^3} - \frac{1}{6\epsilon^2}, \\
 Y(4) &= -\frac{1}{24\epsilon^4} + \frac{1}{24\epsilon^3}, \\
 Y(5) &= \frac{1}{120\epsilon^5} - \frac{1}{120\epsilon^4}, \\
 Y(6) &= -\frac{1}{720\epsilon^6} + \frac{1}{720\epsilon^5}, \\
 Y(7) &= \frac{1}{5040\epsilon^7} - \frac{1}{5040\epsilon^6}, \\
 Y(8) &= -\frac{1}{40320\epsilon^8} + \frac{1}{40320\epsilon^7}, \\
 &\vdots
 \end{aligned}$$

and so on, in this manner $Y(k)$ for $k \geq 9$ can be easily obtained. Therefore, from (2.2), the first few terms of the series solution is as follows:

$$\begin{aligned}
 y(x) &= \frac{x}{\epsilon} + \left(-\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon}\right)x^2 + \left(\frac{1}{6\epsilon^3} - \frac{1}{6\epsilon^2}\right)x^3 \\
 &+ \left(-\frac{1}{24\epsilon^4} + \frac{1}{24\epsilon^3}\right)x^4 + \left(\frac{1}{120\epsilon^5} - \frac{1}{120\epsilon^4}\right)x^5 \\
 &+ \left(-\frac{1}{720\epsilon^6} + \frac{1}{720\epsilon^5}\right)x^6 + \left(\frac{1}{5040\epsilon^7} - \frac{1}{5040\epsilon^6}\right)x^7 \\
 &+ \left(-\frac{1}{40320\epsilon^8} + \frac{1}{40320\epsilon^7}\right)x^8 + \dots
 \end{aligned} \tag{3.4}$$

As the number of terms involved increases, one can observe that the series solution obtained using differential transform method converges to series expansion of the exact solution (3.2). Note that 20 terms are considered in Eq. (3.4) for the numerical results. Comparison of the numerical results with the exact solution (3.2) for $\epsilon = 1, \epsilon = 0.75, \epsilon = 0.5$ and $\epsilon = 0.25$ is shown in Table 2.

Example 2. Consider the following singularly perturbed Volterra integral equation discussed in Alnasr (2000), Alnasr and Momani (2008) and Ibrahim and Alnasr (1998):

$$\epsilon y(x) = \int_0^x (1 + x - t)[1 + t - y(t)]dt. \tag{3.5}$$

The exact solution of this problem is known as

$$y(x) = x + 1 + \frac{1}{\gamma_1 - \gamma_2} \left[\left(\gamma_2 - 1 + \frac{1}{\epsilon} \right) \exp(\gamma_1 x) - \left(\gamma_1 - 1 + \frac{1}{\epsilon} \right) \exp(\gamma_2 x) \right], \tag{3.6}$$

where the parameters γ_1 and γ_2 are defined as

$$\gamma_1 = \frac{1}{2\epsilon} \left(-1 + \sqrt{1 - 4\epsilon} \right), \quad \gamma_2 = \frac{1}{2\epsilon} \left(-1 - \sqrt{1 - 4\epsilon} \right) \tag{3.7}$$

If we apply differential transform to the Eq. (3.5), by using Table 1, we can obtain the following transformed equation:

$$\begin{aligned}
 Y(k) &= \frac{1}{\epsilon} \left(\frac{\delta(k-1) - \delta(k-3)}{k} + \sum_{k_1=1}^k \frac{1}{k_1} \delta(k-k_1-1) [\delta(k_1-1) \right. \\
 &+ \delta(k_1-2) - Y(k_1-1)] + \frac{1}{k} \sum_{k_1=1}^{k-1} \frac{1}{k_1} Y(k_1) [\delta(k-k_1-2) \\
 &\left. - \delta(k-k_1-1) \right), \quad k \geq 1, Y(0) = 0.
 \end{aligned} \tag{3.8}$$

Consequently, we find

$$\begin{aligned}
 Y(1) &= \frac{1}{\epsilon}, \\
 Y(2) &= -\frac{1}{2\epsilon^2} + \frac{1}{\epsilon}, \\
 Y(3) &= \frac{1}{6\epsilon^3} - \frac{1}{2\epsilon^2} + \frac{1}{6\epsilon}, \\
 Y(4) &= \frac{1}{24\epsilon^6} - \frac{1}{6\epsilon^3} + \frac{1}{8\epsilon^2}, \\
 Y(5) &= \frac{1}{120\epsilon^5} - \frac{1}{24\epsilon^4} + \frac{1}{20\epsilon^3} - \frac{1}{120\epsilon^2}, \\
 Y(6) &= -\frac{1}{720\epsilon^6} + \frac{1}{120\epsilon^5} - \frac{1}{72\epsilon^4} + \frac{1}{180\epsilon^3}, \\
 Y(7) &= \frac{1}{5040\epsilon^7} - \frac{1}{720\epsilon^6} + \frac{1}{336\epsilon^5} - \frac{1}{504\epsilon^4} + \frac{1}{5040\epsilon^3}, \\
 Y(8) &= -\frac{1}{40320\epsilon^8} + \frac{1}{5040\epsilon^7} - \frac{1}{1920\epsilon^6} + \frac{1}{2016\epsilon^5} - \frac{1}{8064\epsilon^4}, \\
 &\vdots
 \end{aligned}$$

and so on, in this manner $Y(k)$ for $k \geq 9$ can be easily obtained. Therefore, from (2.2), the first few terms of the series solution are as follows:

Table 2 Numerical results compared to exact solution for Example 1.

x	$\epsilon = 1.0$		$\epsilon = 0.75$		$\epsilon = 0.5$		$\epsilon = 0.25$	
	$y(x)_{Approx}$	$y(x)_{Exact}$	$y(x)_{Approx}$	$y(x)_{Exact}$	$y(x)_{Approx}$	$y(x)_{Exact}$	$y(x)_{Approx}$	$y(x)_{Exact}$
0.0	0.0	0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.1	0.1	0.131207	0.131207	0.190635	0.190635	0.347260	0.347260
0.2	0.2	0.2	0.258518	0.258518	0.364840	0.364840	0.613003	0.613003
0.3	0.3	0.3	0.382420	0.382420	0.525594	0.525594	0.824104	0.824104
0.4	0.4	0.4	0.503338	0.503338	0.675336	0.675336	0.998578	0.998578
0.5	0.5	0.5	0.621646	0.621646	0.816060	0.816060	1.148500	1.148500
0.6	0.6	0.6	0.737668	0.737668	0.949403	0.949403	1.281960	1.281960
0.7	0.7	0.7	0.851690	0.851690	1.076700	1.076700	1.404390	1.404390
0.8	0.8	0.8	0.963962	0.963962	1.199050	1.199050	1.519430	1.519430
0.9	0.9	0.9	1.074700	1.074700	1.317350	1.317350	1.629510	1.629510
1.0	1.0	1.0	1.184100	1.184100	1.432330	1.432330	1.736260	1.736260

Table 3 Numerical results compared to exact solution for Example 2.

x	ε = 1.0		ε = 0.75		ε = 0.5	
	y(x) _{Approx}	y(x) _{Exact}	y(x) _{Approx}	y(x) _{Exact}	y(x) _{Approx}	y(x) _{Exact}
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.104833	0.104833	0.137510	0.137510	0.199683	0.199683
0.2	0.218669	0.218669	0.282340	0.282340	0.397589	0.397589
0.3	0.340519	0.340519	0.433015	0.433015	0.592269	0.592269
0.4	0.469413	0.469413	0.588175	0.588175	0.782594	0.782594
0.5	0.604405	0.604405	0.746572	0.746572	0.967719	0.967719
0.6	0.744584	0.744584	0.907074	0.907074	1.147046	1.147046
0.7	0.889072	0.889072	1.068664	1.068664	1.320191	1.320191
0.8	1.037037	1.037037	1.230437	1.230437	1.486949	1.486949
0.9	1.187692	1.187692	1.391602	1.391602	1.647272	1.647272
1.0	1.340300	1.340300	1.551470	1.551470	1.801234	1.801234

$$\begin{aligned}
 y(x) = & \frac{x}{\varepsilon} + \left(-\frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}\right)x^2 + \left(\frac{1}{6\varepsilon^3} - \frac{1}{2\varepsilon^2} + \frac{1}{6\varepsilon}\right)x^3 \\
 & + \left(-\frac{1}{24\varepsilon^6} + \frac{1}{6\varepsilon^3} - \frac{1}{8\varepsilon^2}\right)x^4 \\
 & + \left(\frac{1}{120\varepsilon^5} - \frac{1}{24\varepsilon^4} + \frac{1}{20\varepsilon^3} - \frac{1}{120\varepsilon^2}\right)x^5 \\
 & + \left(-\frac{1}{720\varepsilon^6} + \frac{1}{120\varepsilon^5} - \frac{1}{72\varepsilon^4} + \frac{1}{180\varepsilon^3}\right)x^6 \\
 & + \left(\frac{1}{5040\varepsilon^7} - \frac{1}{720\varepsilon^6} + \frac{1}{336\varepsilon^5} - \frac{1}{504\varepsilon^4} + \frac{1}{5040\varepsilon^3}\right)x^7 \\
 & + \left(-\frac{1}{40320\varepsilon^8} + \frac{1}{5040\varepsilon^7} - \frac{1}{1920\varepsilon^6} + \frac{1}{2016\varepsilon^5} - \frac{1}{8064\varepsilon^4}\right)x^8 \\
 & + \dots
 \end{aligned}
 \tag{3.9}$$

As the number of terms involved increases, one can observe that the series solution obtained using differential transform method converges to series expansion of the exact solution (3.6). Note that 11 terms are considered in Eq. (3.9) for the numerical results. Comparison of the numerical results with the exact solution (3.6) for ε = 1, ε = 0.75 and ε = 0.5 are shown in Table 3.

Example 3. We consider the following nonlinear singularly perturbed Volterra Integral Equation discussed in Alnasr (2000) and Alnasr and Momani (2008):

$$\varepsilon y(x) = \int_0^x e^{x-t} [y^2(t) - 1] dt,
 \tag{3.10}$$

which has the exact solution

$$y(x) = \frac{2(1 - e^{\gamma x})}{\varepsilon(\gamma - 1)e^{\gamma x} + \gamma + 1}, \quad \gamma = \frac{1}{\varepsilon} \sqrt{4 + \varepsilon^2}.
 \tag{3.11}$$

One can see that the differential transform of Eq. (3.10) can be evaluated by using Table 1 as follows:

$$\begin{aligned}
 Y(k) = & \frac{1}{\varepsilon} \left(\sum_{k_3=1}^k \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} \frac{1}{k_3} \frac{(-1)^{k_1-1}}{(k_1-1)!} Y(k_2-k_1) Y(k_3-k_2) \right. \\
 & \times \left. \frac{1}{(k-k_3)!} - \sum_{k_1=1}^k \frac{1}{k_1} \frac{1}{(k-k_1)!} \frac{(-1)^{k_1-1}}{(k_1-1)!} \right), \\
 k \geq 1, Y(0) = & 0.
 \end{aligned}
 \tag{3.12}$$

Consequently, we find

$$\begin{aligned}
 Y(1) = & -\frac{1}{\varepsilon}, \\
 Y(2) = & -\frac{1}{2\varepsilon}, \\
 Y(3) = & \frac{1}{3\varepsilon^3} - \frac{1}{6\varepsilon}, \\
 Y(4) = & \frac{1}{3\varepsilon^3} - \frac{1}{24\varepsilon}, \\
 Y(5) = & -\frac{2}{15\varepsilon^5} + \frac{11}{60\varepsilon^3} - \frac{1}{120\varepsilon}, \\
 Y(6) = & -\frac{17}{90\varepsilon^5} + \frac{13}{180\varepsilon^3} - \frac{1}{720\varepsilon}, \\
 Y(7) = & \frac{17}{315\varepsilon^7} - \frac{1}{7\varepsilon^5} + \frac{19}{840\varepsilon^3} - \frac{1}{5040\varepsilon}, \\
 Y(8) = & \frac{31}{315\varepsilon^7} - \frac{8}{105\varepsilon^5} + \frac{1}{168\varepsilon^3} - \frac{1}{40320\varepsilon}, \\
 & \vdots
 \end{aligned}$$

and so on, in this manner Y(k) for k ≥ 9 can be easily obtained. Therefore, from (2.2), the solution of the integral equation (3.10) is given as

$$\begin{aligned}
 y(x) = & -\frac{x}{\varepsilon} - \frac{1}{2\varepsilon}x^2 + \left(\frac{1}{3\varepsilon^3} - \frac{1}{6\varepsilon}\right)x^3 + \left(\frac{1}{3\varepsilon^3} - \frac{1}{24\varepsilon}\right)x^4 \\
 & + \left(-\frac{2}{15\varepsilon^5} + \frac{11}{60\varepsilon^3} - \frac{1}{120\varepsilon}\right)x^5 \\
 & + \left(-\frac{17}{90\varepsilon^5} + \frac{13}{180\varepsilon^3} - \frac{1}{720\varepsilon}\right)x^6 \\
 & + \left(\frac{17}{315\varepsilon^7} - \frac{1}{7\varepsilon^5} + \frac{19}{840\varepsilon^3} - \frac{1}{5040\varepsilon}\right)x^7 \\
 & + \left(\frac{31}{315\varepsilon^7} - \frac{8}{105\varepsilon^5} + \frac{1}{168\varepsilon^3} - \frac{1}{40320\varepsilon}\right)x^8 + \dots
 \end{aligned}
 \tag{3.13}$$

The evolution results for the exact solution (3.11) and the approximate solution obtained using differential transform method, for different values of ε, are shown in Figs. 1–3. Note that 11 terms are considered in Eq. (3.13) for the numerical results.

Example 4. We consider the following singularly perturbed Volterra integral equation (Kauthen, 1997):

$$\varepsilon y(x) = \sin x - \int_0^x y(t) dt,
 \tag{3.14}$$

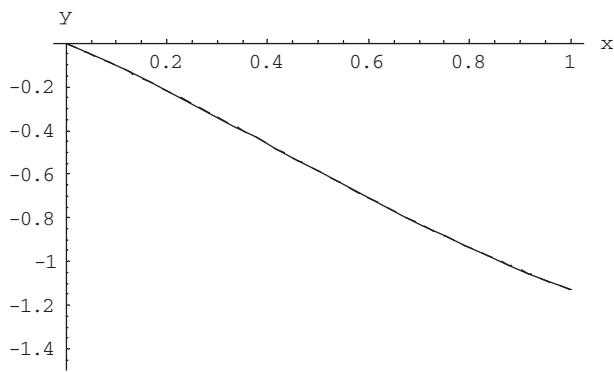


Figure 1 Plots of Eq. (3.10) when $\varepsilon = 1$. Exact solution (—); the approximate solution (---).

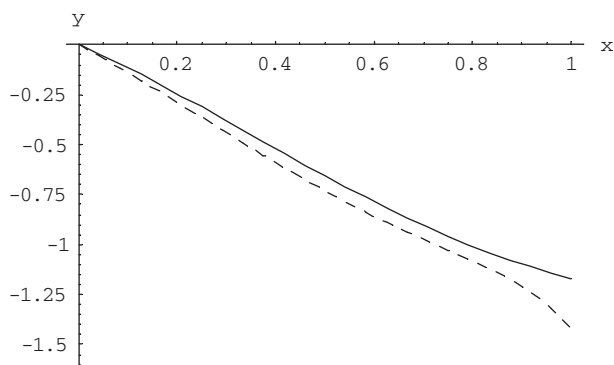


Figure 2 Plots of Eq. (3.10) when $\varepsilon = 0.75$. Exact solution (—); the approximate solution (---).

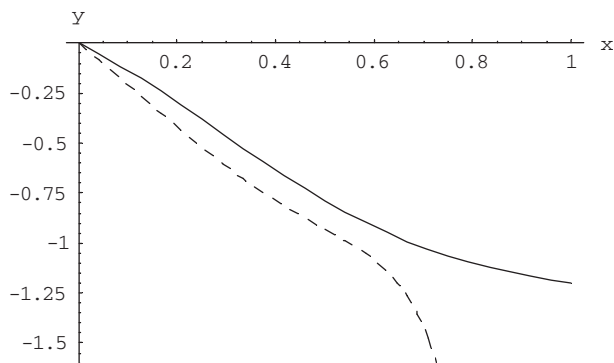


Figure 3 Plots of Eq. (3.10) when $\varepsilon = 0.5$. Exact solution (—); the approximate solution (---).

which has the exact solution

$$y(x) = \frac{1}{1 + \varepsilon^2} (\cos x + \varepsilon \sin x - e^{-\frac{x}{\varepsilon}}). \tag{3.15}$$

According to the operations of differential transformation given in Table 1, the following recurrence relation is obtained:

$$Y(k) = \frac{1}{\varepsilon} \left[\frac{1}{k!} \sin\left(\frac{\pi k}{2}\right) - \frac{Y(k-1)}{k} \right]. \tag{3.16}$$

Utilizing the recurrence relation in Eq. (3.16), we find

$$\begin{aligned} Y(1) &= \frac{1}{\varepsilon}, \\ Y(2) &= -\frac{1}{2\varepsilon^2}, \\ Y(3) &= \frac{1}{6\varepsilon^3} - \frac{1}{6\varepsilon}, \\ Y(4) &= -\frac{1}{24\varepsilon^4} + \frac{1}{24\varepsilon^2}, \\ Y(5) &= \frac{1}{120\varepsilon^5} - \frac{1}{120\varepsilon^3} + \frac{1}{120\varepsilon}, \\ Y(6) &= -\frac{1}{720\varepsilon^6} + \frac{1}{720\varepsilon^4} - \frac{1}{720\varepsilon^2}, \\ Y(7) &= \frac{1}{5040\varepsilon^7} - \frac{1}{5040\varepsilon^5} + \frac{1}{5040\varepsilon^3} \\ &\quad - \frac{1}{5040\varepsilon}, \\ Y(8) &= -\frac{1}{40320\varepsilon^8} + \frac{1}{40320\varepsilon^6} - \frac{1}{40320\varepsilon^4} + \frac{1}{40320\varepsilon^2}, \\ &\quad \vdots \end{aligned}$$

and so on, in this manner $Y(k)$ for $k \geq 9$ can be easily obtained. Therefore, from (2.2), the first few terms of the series solution are as follows:

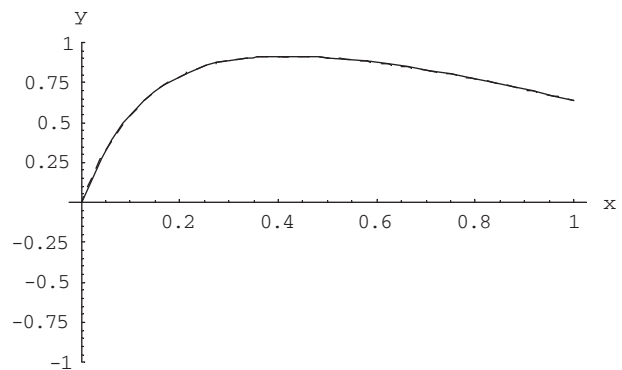


Figure 4 Plots of Eq. (3.14) when $\varepsilon = 2^{-3}$. Exact solution (—); the approximate solution (---).

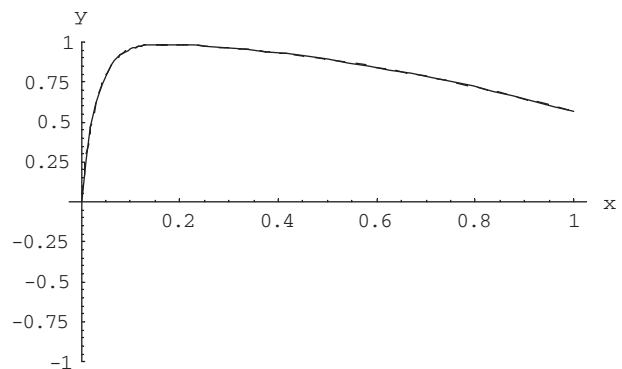


Figure 5 Plots of Eq. (3.14) when $\varepsilon = 2^{-5}$. Exact solution (—); the approximate solution (---).

$$\begin{aligned}
y(x) = & \frac{x}{\varepsilon} - \frac{1}{2\varepsilon}x^2 + \left(\frac{1}{6\varepsilon^3} - \frac{1}{6\varepsilon}\right)x^3 \\
& + \left(-\frac{1}{24\varepsilon^4} + \frac{1}{24\varepsilon^2}\right)x^4 \\
& + \left(\frac{1}{120\varepsilon^5} - \frac{1}{120\varepsilon^3} + \frac{1}{120\varepsilon}\right)x^5 \\
& + \left(-\frac{1}{720\varepsilon^6} + \frac{1}{720\varepsilon^4} - \frac{1}{720\varepsilon^2}\right)x^6 \\
& + \left(\frac{1}{5040\varepsilon^7} - \frac{1}{5040\varepsilon^5} + \frac{1}{5040\varepsilon^3} - \frac{1}{5040\varepsilon}\right)x^7 \\
& + \left(-\frac{1}{40320\varepsilon^8} + \frac{1}{40320\varepsilon^6} - \frac{1}{40320\varepsilon^4} + \frac{1}{40320\varepsilon^2}\right)x^8 \\
& + \dots
\end{aligned} \tag{3.17}$$

The evolution results for the exact solution (3.15) and the approximate solution obtained using differential transform method, for $\varepsilon = 2^{-3}$ and $\varepsilon = 2^{-5}$, are shown in Figs. 4 and 5. Note that 90 terms are considered in Eq. (3.17) for the numerical results.

4. Conclusion

In this study, Differential transform method is successfully applied to singularly perturbed Volterra integral equations. A symbolic calculation software package, MATHEMATICA is used for all calculations. All the computations show that the approximate solutions are perfectly identical to the exact solutions. Also, the work emphasized our belief that the method is a reliable technique to handle these types of problems. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, discretization or restrictive assumptions.

References

Alnasr, M.H., 2000. Modified Multilag methods for singularly perturbed Volterra integral equations. *Int. J. Comput. Math.* 75 (1), 221–233.

- Alnasr, M.H., Momani, Sahaher, 2008. Application of Homotopy perturbation method to singularly perturbed Volterra integral equations. *J. Appl. Sci.* 8 (6), 1073–1078.
- Alnasr, M.H., 1997. Numerical Treatment of Singularly Perturbed Volterra Integral Equations. Ph.D. Thesis. Egypt.
- Angell, J.S., Olmstead, W.E., 1987. Singularly perturbed Volterra integral equations. *Siam J. Appl. Math.* 47 (1), 1–14.
- Arıkoğlu, A., Özkol, İ., 2005. Solution of boundary value problems for integro-differential equations by using differential transform method. *Appl. Math. Comput.* 168, 1145–1158.
- Ayaz, F., 2004. Solutions of the system of differential equations by differential transform method. *Appl. Math. Comput.* 147, 547–567.
- Bildik, N., Konuralp, A., Bek, F., Küçükarslan, S., 2006. Solution of different type of the partial differential equation by differential transform method and Adomian's decomposition method. *Appl. Math. Comput.* 172, 551–567.
- Brunner, H., Van Der Houwen, P.J., 1986. *The Numerical Solution of Volterra Equations*. CWI, Amsterdam.
- Ertürk, V.S., Momani, S., 2007. A reliable algorithm for solving tenth-order boundary value problems. *Numer. Algorithms* 44 (2), 147–158.
- Ertürk, V.S., Momani, S., 2007. Comparing numerical methods for solving fourth-order boundary value problems. *Appl. Math. Comput.* 188, 1963–1968.
- Hassan, I.H.A.H., 2008. Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value problems. *Chaos Solitons Fractals* 36 (1), 53–65.
- Ibrahim, M.A.-K., Alnasr, M.H., 1998. Quadrature methods with regularization technique for singularly perturbed Volterra integral equations. *Int. J. Comput. Math.* 68, 285–299.
- Kauthen, J.P., 1997. A survey of singularly perturbed Volterra equations. *Appl. Numer. Math.* 24, 95–114.
- Lange, C.G., Smith, D.R., 1988. Singular perturbation analysis of integral equations. *Stud. Appl. Math.* 79 (11), 1–63.
- Liu, H., Song, Y., 2007. Differential transform method applied to high index differential-algebraic equations. *Appl. Math. Comput.* 184, 748–753.
- Pukhov, G.E., 1980. *Differential Transforms of Functions and Equations*. Naukova Dumka (in Russian), Kiev.
- Zhou, J., 1986. *Differential Transformation and Its Application for Electrical Circuits*. Huazhong University Press (in Chinese).