



Research article

Development of spreading symmetric two-waves motion for a family of two-mode nonlinear equations

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ABSTRACT

In this work, a functional operator extracted from Korsunsky's technique is used to produce new two-mode nonlinear equations. These new equations describe the motion of two directional solitary-waves overlapping with an increasing phase-velocity and affected by two factors labeled as the dispersion and nonlinearity coefficients. To investigate the dynamics of this two-mode family, we construct the two-mode KdV–Burgers–Kuramoto equation (TMKKBK) and two-mode Hirota–Satsuma model (TMHS). Two efficient schemes are used to assign the necessary constraints for existence of solutions and to extract them. The role of the phase-velocity on the motion of the obtained two-wave solutions is investigated graphically. Finally, all the obtained solutions are categorized according to their physical shapes.

1. Introduction

Two-mode nonlinear equations are characterized by three factors; nonlinearity-parameter, dispersion-parameter and a phase-velocity parameter. Physically, these equations describe the propagation of solitary-waves spreading in the shape of two overlapping waves which is dependent on the phase velocity.

Collision or overlapping of solitary waves due to distinct wave modes have been investigated in [1]. It has been observed that such overlapping may occur when the phase speeds for the phase-locked traveling waves and overtaking waves are close to each other [1]. A temporal-second-order KdV equation, in which these overlapping can be seen, has been proposed as [2]

$$W_{\tau\tau} + (c_1 + c_2)W_{\chi\tau} + c_1c_2W_{\chi\chi} + ((\alpha_1 + \alpha_2)\frac{\partial}{\partial\tau} + (\alpha_1c_2 + \alpha_2c_1)\frac{\partial}{\partial\chi})WW_{\chi} + ((\beta_1 + \beta_2)\frac{\partial}{\partial\tau} + (\beta_1c_2 + \beta_2c_1)\frac{\partial}{\partial\chi})W_{\chi\chi\chi} = 0, \quad (1.1)$$

where χ and τ are the scaled-space and time coordinates, $W(\chi, \tau)$ is the height of the water's free surface above flat bottom, c_1 and c_2 are the phase velocities, α_1 and α_2 are the nonlinearity parameters, and β_1 and β_2 refer to the dispersion parameters [3].

In [2], Korsunsky considered the following transformations

$$\begin{aligned} x &= \frac{\chi - \frac{c_1+c_2}{2}\tau}{\sqrt{\beta_1 + \beta_2}}, \\ t &= \frac{\tau}{\sqrt{\beta_1 + \beta_2}}, \\ w &= (\alpha_1 + \alpha_2)W, \end{aligned} \quad (1.2)$$

and the following restrictions

$$\begin{aligned} \left| \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} \right| &\leq 1, \\ \left| \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} \right| &\leq 1, \\ c_2 &\leq c_1. \end{aligned} \quad (1.3)$$

By using (1.2) and (1.3), Equation (1.1) is converted into

$$w_{tt} - s^2w_{xx} + \left(\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x} \right) ww_x + \left(\frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x} \right) w_{xxx} = 0. \quad (1.4)$$

Now, $w = w(x, t)$ is the field function, α , β , respectively, are the nonlinearity and dispersion parameters dominated by the value of 1, and s is the interaction or the overlapping phase velocity. Note that, in case

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of $s = 0$, no interaction, and integrating once with respect to the time t , (1.4) is reduced to the standard KdV which possesses a single-moving wave. Equation (1.4) is known as two-mode KdV because it possesses the motion of two waves moving at the same time with interaction phase velocity of value s .

Different types of two-mode equations have been established and investigated. Wazwaz [4] extracted multiple-kink solutions of two-mode Sharma–Tasso–Olver equation and two-mode fourth-order Burgers' by means of the simplified Hirota's method. Also, Wazwaz established and studied the solution for two-wave mode Kadomtsev–Petviashvili, two-mode fifth-order KdV and two-mode higher-order modified KdV equations [5, 6, 7]. In [8, 9, 10], an implementation of bilinear method is applied for the two-mode coupled Burgers' equation, two-mode coupled KdV, and two-mode coupled modified KdV. In [11, 12], the authors revisited the two-mode KdV and two-mode Sharma–Tasso–Olver for more new solitary wave solutions. More recently, the authors of [13, 14, 15] investigated the two-mode KdV–Burgers' equation, two-mode Kuramoto–Sivashinsky and two-mode higher-order Boussinesq–Burger system. They recommended some analysis regarding the dispersive factor β and the velocity factor s .

It should be noted here that all the investigations and studies regarding the two-mode equations are mainly based on Korsunsky proposal scheme. We expect that other schemes will be developed and play more significant roles in the construction of two-mode models. Therefore, the Korsunsky method for deriving two-mode equations has the following scaled form

$$w_{tt} - s^2 w_{xx} + \left(\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x}\right) N(w, w_x, \dots) + \left(\frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x}\right) L(w_{kx}) = 0, \quad k \geq 2. \tag{1.5}$$

$N(w, w_x, \dots)$ and $L(w_{kx})$ are the nonlinear and the linear terms of the model. Using (1.5), we proceed to introduce two new two-mode models.

1.1. Two-mode KBK equation

The standard KBK equation reads [16]

$$u_t + uu_x + \gamma_1 u_{xx} + \gamma_2 u_{xxx} + \gamma_3 u_{xxxx} = 0, \tag{1.6}$$

where the parameters γ_1 , γ_2 , and γ_3 are, respectively, dissipation, dispersion and instability. In this equation, $N = uu_x$ and $L = \gamma_1 u_{xx} + \gamma_2 u_{xxx} + \gamma_3 u_{xxxx}$. Using the scale (1.5), the two-mode KdV–Burgers–Kuramoto equation (TMKKBK) has the form

$$u_{tt} - s^2 u_{xx} + \left(\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x}\right) \{uu_x\} + \left(\frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x}\right) \{\gamma_1 u_{xx} + \gamma_2 u_{xxx} + \gamma_3 u_{xxxx}\} = 0. \tag{1.7}$$

The field function $u(x, t)$ stands for the “height of the water’s free surface above a flat bottom”, α and β are the dispersion and nonlinearity parameters and s is a positive integer defined as the phase-velocity.

1.2. Two-mode HS equation

The Hirota Satsuma (HS) under investigation is given by the following version [17]

$$v_t = \frac{1}{4} v_{xxx} + 3vv_x - 6w w_x, \\ w_t = -\frac{1}{2} w_{xxx} - 3vw_x, \tag{1.8}$$

which describes overlapping of two long-waves with distinct dispersion relations. Considering the operator defined above, the two-mode Hirota Satsuma (TMHS) model can be defined in the following form

$$0 = v_{tt} - s^2 v_{xx} + \left(\frac{\partial}{\partial t} - \alpha_1 s \frac{\partial}{\partial x}\right) \{-3vv_x + 6w w_x\} - \frac{1}{4} \left(\frac{\partial}{\partial t} - \beta_1 s \frac{\partial}{\partial x}\right) \{v_{xxx}\}, \\ 0 = w_{tt} - s^2 w_{xx} + 3 \left(\frac{\partial}{\partial t} - \alpha_2 s \frac{\partial}{\partial x}\right) \{vw_x\} + \frac{1}{2} \left(\frac{\partial}{\partial t} - \beta_2 s \frac{\partial}{\partial x}\right) \{w_{xxx}\}. \tag{1.9}$$

There are many solitary ansatz methods are available in the literature, some of these in some cases provide the same solutions, while others give different physical type-solutions. Such well-known methods are, trigonometric-function method, Jacobi-elliptic-function method, (G'/G) -expansion, tanh-expansion, bilinear method, Jacobi elliptic expansion method and others [18, 19, 20, 21, 22, 23, 24]. These methods are seen to have the form of finite series in terms of trigonometric/hyperbolic functions. In this study, we adopt the tanh-coth expansion method and Kudryashov method to provide a physical analysis on the dynamics of such new models. The proposed methods provide more general type solitary wave solutions which are necessary to explore the dynamics of the physical phenomena. Another merit of these methods that it deal with only one coordinate which is a linear combination of all involved coordinate-axes of the origin equation. Therefore, the degree of freedom of the derived solution is much less than other existing methods which depend on all its coordinate parameters. One more advantage, that these methods are straightforward and precise, it requires no linearization as Hirota method and no sophisticated assumptions, and hence, it provides efficient solutions.

2. Solutions of two-mode KBK

In this section, we establish solitary wave solutions for TMKKBK given in (1.7). We transform the proposed model using the wave transform $\zeta = x - ct$ into the following reduced ordinary differential equation (ODE)

$$(c^2 - s^2)u - \frac{1}{2}(c + \alpha s)u^2 - (c + \beta s)(\gamma_1 u' + \gamma_2 u'' + \gamma_3 u''') = 0. \tag{2.10}$$

2.1. Tanh-coth solution: TMKKBK

In this method, we write the solution of (2.10) as a finite series of tanh-function [25, 26, 27, 28], i.e.,

$$u(\zeta) = \sum_{i=0}^m a_i Y^i, \tag{2.11}$$

where

$$Y = \tanh(\mu\zeta). \tag{2.12}$$

To find m , we compare the linear term u''' with the nonlinear term u^2 which gives $m + 3 = 2m$. Thus, $m = 3$. Accordingly, (2.11) takes the following form

$$u(\zeta) = a_0 + a_1 \tanh(\mu\zeta) + a_2 \tanh^2(\mu\zeta) + a_3 \tanh^3(\mu\zeta). \tag{2.13}$$

Inserting (2.13) into (2.10) and applying the relation “ $\text{sech}^2(\mu\zeta) = 1 - \tanh^2(\mu\zeta)$ ”, produces a finite series of $\tanh(\mu\zeta)$. The coefficients of same powers of \tanh are identical to zero. By solving the resulting system for the unknown parameters, we arrive at the following outcomes

$$\alpha = \beta = \pm 1, \\ a_0 = \text{free}, \quad a_1 = \frac{-15\mu(\gamma_2^2 + 16\gamma_3(38\gamma_3\mu^2 - \gamma_1))}{76\gamma_3}, \quad a_2 = 0, \quad a_3 = 120\gamma_3\mu^3, \\ c = -s\beta, \quad \mu = \text{free}. \tag{2.14}$$

Therefore, the two-waves solutions for TMKKBK is

$$u(x, t) = a_0 - \frac{15\mu(\gamma_2^2 + 16\gamma_3(38\gamma_3\mu^2 - \gamma_1))}{76\gamma_3} \tanh(\mu(x \pm t)) \\ + 120\gamma_3\mu^3 \tanh^3(\mu(x \pm t)). \tag{2.15}$$

We should note here that the free parameters a_0 and μ are to be chosen such that the quantity $u(x, t)$ is always positive. Fig. 1, shows the interaction of these two-waves (2.15) spreading with increasing angular-phase at increasing s . The shape of the depicted solution is a combination of kink-type and soliton-type. Also, in Fig. 2, 2D profile-solutions for

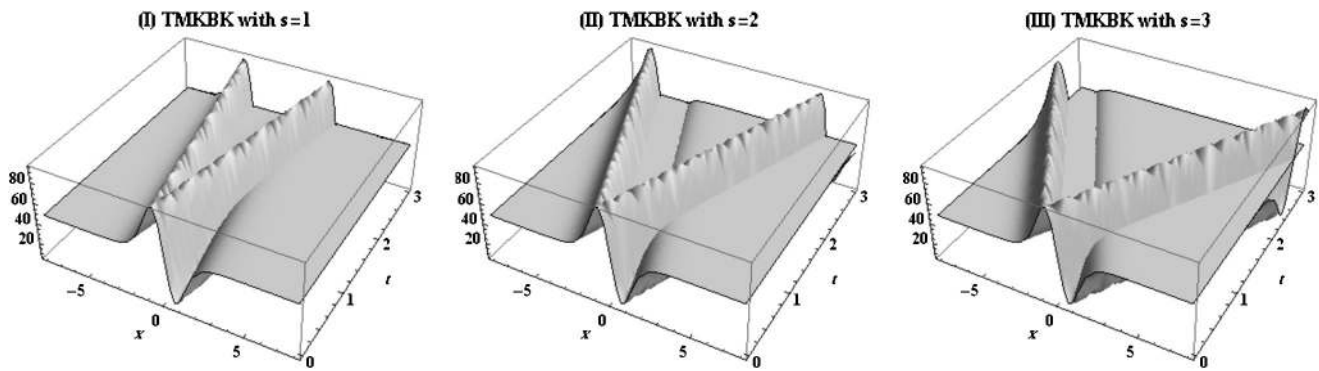


Fig. 1. Overlapping of two-shock waves obtained for TMKKB given in (2.15) by means of tanh-expansion method. (I) $s = 1$. (II) $s = 2$. (III) $s = 3$. Where $\gamma_1 = \gamma_2 = \gamma_3 = \mu = 1$ and $a_0 = 50$.

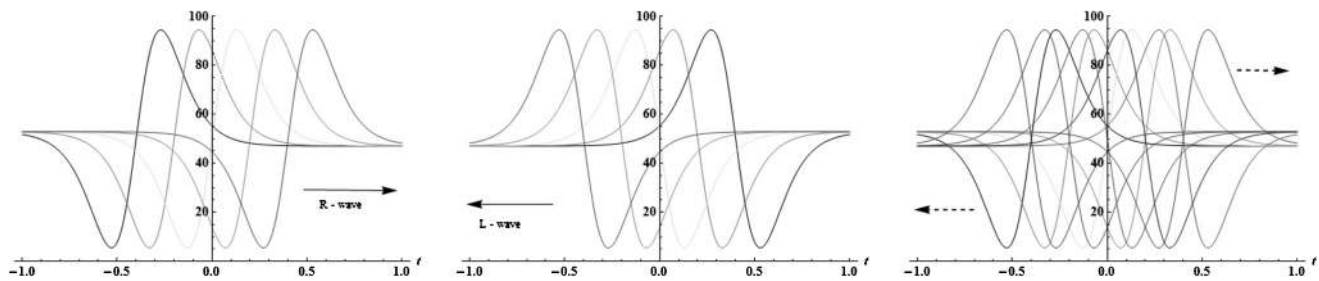


Fig. 2. 2D Profile solutions for TMKKB depicted in (2.15). Right-shock, left-shock and right-left-shock waves. Where $\gamma_1 = \gamma_2 = \gamma_3 = \mu = a_0 = 1$ and $s = 5$.

the TMKKB are given and can be viewed as right-shock, left-shock and right-left-shock waves.

In (2.12)–(2.13), if the tanh-function is replaced by coth-function, we obtain the same above findings but leads to a singular two-waves solutions for TMKKB given as

$$u(x, t) = a_0 - \frac{15\mu(\gamma_2^2 + 16\gamma_3(38\gamma_3\mu^2 - \gamma_1))}{76\gamma_3} \coth(\mu(x \pm t)) + 120\gamma_3\mu^3 \coth^3(\mu(x \pm t)). \tag{2.16}$$

The physical shape of this new solution will be a combination of kink and soliton but with singularities through the characteristic lines $x = \pm t$.

2.2. Kudryashov-expansion solution: TMKKB

Kudryashov method [29, 30, 31, 32] assumes the solution of (2.10) as a finite series in the variable Z

$$u(\zeta) = \sum_{i=0}^n b_i Z^i. \tag{2.17}$$

The variable $Z = Z(\zeta)$ is the solution of the nonlinear differential equation

$$Z' = \mu Z(Z - 1). \tag{2.18}$$

Applying the separable method on (2.18) gives

$$Z = Z(\zeta) = \frac{1}{1 + de^{\mu\zeta}}. \tag{2.19}$$

Performing the balance-procedure gives $n = 3$ and accordingly we write (2.17) as

$$u(\zeta) = b_0 + b_1 Z + b_2 Z^2 + b_3 Z^3. \tag{2.20}$$

Differentiating both (2.18) and (2.20) implicitly, leads to

$$\begin{aligned} Z'' &= \mu^2 Z(Z - 1)(2Z - 1), \\ Z''' &= \mu^3 Z(Z - 1)(6Z^2 - 6Z + 1) \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} u'(z) &= b_1 Z' + 2b_2 Z Z' + 3b_3 Z^2 Z', \\ u''(z) &= b_1 Z'' + 2b_2(Z Z'' + (Z')^2) + 3b_3(2Z(Z')^2 + Z^2 Z''), \\ u'''(z) &= b_1 Z''' + 2b_2(Z Z''' + 3Z' Z'') + 3b_3(2(Z')^3 + 6Z Z' Z'' + Z^2 Z'''). \end{aligned} \tag{2.22}$$

Now, we substitute (2.18) through (2.22) in (2.10) to get a finite series in Z whose coefficients are identical to zero. To be able solving the resulting system, we impose the constraint relation that $\gamma_1 = 47\gamma_3\mu^2$ and $\gamma_2 = 36\gamma_3\mu$. Accordingly we get

$$\begin{aligned} \alpha &= \beta = k : -1 \leq k \leq 1, \\ b_0 &= b_1 = b_2 = 0, \\ b_3 &= -120\gamma_3\mu^3, \\ c &= -30\gamma_3\mu^3 \pm \sqrt{s^2 - 60k s \gamma_3 \mu^3 + 900\gamma_3^2 \mu^6}. \end{aligned} \tag{2.23}$$

Therefore, a new solution of TMKKB is

$$u(x, t) = - \frac{120\gamma_3\mu^3}{\left(1 + de^{\mu x + \left(30\gamma_3\mu^3 \pm \sqrt{s^2 - 60k s \gamma_3 \mu^3 + 900\gamma_3^2 \mu^6}\right)t}\right)^3}. \tag{2.24}$$

Fig. 3, shows the overlapping of these two-kink-waves (2.24) spreading with increasing angular-phase upon increasing the phase velocity s .

3. Solutions of two-mode Hirota–Satsuma model

In this section, we seek soliton solutions of the TMHS given in (1.9). We convert the proposed model through the wave transform $\zeta = x - \lambda t$ into a system of ordinary differentials

$$\begin{aligned} 0 &= (\lambda^2 - s^2)v - \frac{1}{2}(\lambda + \alpha_1 s)(-3v^2 + 6w^2) + \frac{1}{4}(\lambda + \beta_1 s)v'', \\ 0 &= (\lambda^2 - s^2)w' - 3(\lambda + \alpha_2 s)vw' - \frac{1}{2}(\lambda + \beta_2 s)w'''. \end{aligned} \tag{3.25}$$

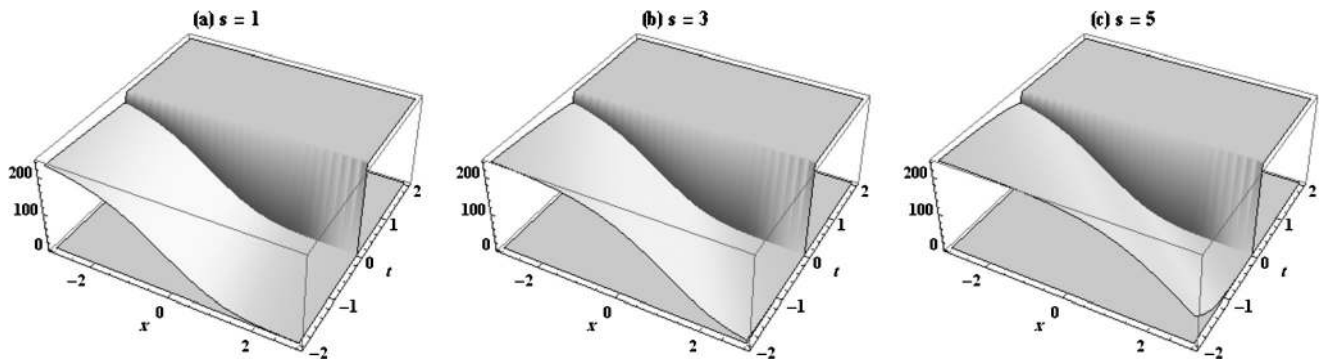


Fig. 3. Overlapping of two-shock waves obtained for TMKBK given in (2.24) by means of Kudryashov-expansion method. (a) $s = 1$. (b) $s = 3$. (c) $s = 5$. Where $\mu = 1$, γ_3 , $d = 0.7$, $k = 0.3$.

3.1. Tanh-coth solution: TMHS

Adopting the finite tanh-function series solution, we let

$$v(\zeta) = \sum_{i=0}^{m_1} b_i \tanh^i(\mu\zeta),$$

$$w(\zeta) = \sum_{i=0}^{m_2} c_i \tanh^i(\mu\zeta). \tag{3.26}$$

Performing the balance procedure between v'' versus w^2 and w''' versus vw' , we get $m_1 + 2 = 2m_2$ and $m_2 + 3 = m_1 + m_2 + 1$. Thus, $m_1 = m_2 = 2$. Then, we substitute both v and w from (3.26) in the reduced system (3.25). The resulting equations are polynomials with powers of $\tanh(\mu\zeta)$ whose coefficients are set to zero. The resulting system is undetermined with the unknowns μ , λ , b_0 , b_1 , b_2 , c_0 , c_1 , c_2 . We observed that this system can not be solved unless we impose some more constraint relations. We require only one convenient condition which is

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = k, \quad |k| < 1. \tag{3.27}$$

Therefore, by solving the modified algebraic system, we arrive at the following outcomes

$$b_2 = -2\mu^2, \quad b_1 = 0, \quad b_0 = \frac{1}{15}(20 - \sqrt{10})\mu^2,$$

$$c_2 = -\mu^2, \quad c_1 = 0, \quad c_0 = \frac{2}{15}(5 - \sqrt{10})\mu^2,$$

$$\lambda = -\frac{\mu^2 \pm \sqrt{10s^2 - 2\sqrt{10}ks\mu^2 + \mu^4}}{\sqrt{10}}. \tag{3.28}$$

Now, putting these values in (3.26), will produce two-soliton-waves solutions to TMHS,

$$v(x, t) = \frac{1}{15}(20 - \sqrt{10})\mu^2$$

$$- 2\mu^2 \tanh^2 \left[\mu \left(x + t \left(\frac{\mu^2 \pm \sqrt{10s^2 - 2\sqrt{10}ks\mu^2 + \mu^4}}{\sqrt{10}} \right) \right) \right],$$

$$w(x, t) = \frac{2}{15}(5 - \sqrt{10})\mu^2$$

$$- \mu^2 \tanh^2 \left[\mu \left(x + t \left(\frac{\mu^2 \pm \sqrt{10s^2 - 2\sqrt{10}ks\mu^2 + \mu^4}}{\sqrt{10}} \right) \right) \right]. \tag{3.29}$$

Moreover, by replacing the parameter μ by $i\gamma$, then $\tanh(\mu x) = i \tan(\gamma x)$. Accordingly, from (3.29), we obtain new two-periodic-waves solution for the TMHS model,

$$v(x, t) = \frac{1}{15}(\sqrt{10} - 20)\gamma^2$$

$$- 2\gamma^2 \tan^2 \left[\gamma \left(x + t \left(\frac{-\gamma^2 \pm \sqrt{10s^2 + 2\sqrt{10}ks\gamma^2 + \gamma^4}}{\sqrt{10}} \right) \right) \right],$$

$$w(x, t) = \frac{2}{15}(\sqrt{10} - 5)\gamma^2$$

$$- \gamma^2 \tan^2 \left[\gamma \left(x + t \left(\frac{-\gamma^2 \pm \sqrt{10s^2 + 2\sqrt{10}ks\gamma^2 + \gamma^4}}{\sqrt{10}} \right) \right) \right]. \tag{3.30}$$

In (3.26), if the tanh-function is replaced by coth-function, one more solution of type singular-soliton will be reported as

$$v(x, t) = \frac{1}{15}(20 - \sqrt{10})\mu^2$$

$$- 2\mu^2 \coth^2 \left[\mu \left(x + t \left(\frac{\mu^2 \pm \sqrt{10s^2 - 2\sqrt{10}ks\mu^2 + \mu^4}}{\sqrt{10}} \right) \right) \right],$$

$$w(x, t) = \frac{2}{15}(5 - \sqrt{10})\mu^2$$

$$- \mu^2 \coth^2 \left[\mu \left(x + t \left(\frac{\mu^2 \pm \sqrt{10s^2 - 2\sqrt{10}ks\mu^2 + \mu^4}}{\sqrt{10}} \right) \right) \right]. \tag{3.31}$$

If we replace the parameter μ by $i\gamma$, then $\coth(\mu x) = -i \cot(\gamma x)$. Therefore, from (3.31), we obtain new two-periodic-waves solution for TMHS model,

$$v(x, t) = \frac{1}{15}(\sqrt{10} - 20)\gamma^2$$

$$- 2\gamma^2 \cot^2 \left[\gamma \left(x + t \left(\frac{-\gamma^2 \pm \sqrt{10s^2 + 2\sqrt{10}ks\gamma^2 + \gamma^4}}{\sqrt{10}} \right) \right) \right],$$

$$w(x, t) = \frac{2}{15}(\sqrt{10} - 5)\gamma^2$$

$$- \gamma^2 \cot^2 \left[\gamma \left(x + t \left(\frac{-\gamma^2 \pm \sqrt{10s^2 + 2\sqrt{10}ks\gamma^2 + \gamma^4}}{\sqrt{10}} \right) \right) \right]. \tag{3.32}$$

3.2. Kudryashov-expansion solution: TMHS

The Kudryashov scheme has the same order balance as in tanh-coth technique. Therefore, the suggested solution of (3.25) takes the following form

$$v(\zeta) = a_0 + a_1 Z + a_2 Z^2,$$

$$w(\zeta) = b_0 + b_1 Z + b_2 Z^2, \tag{3.33}$$

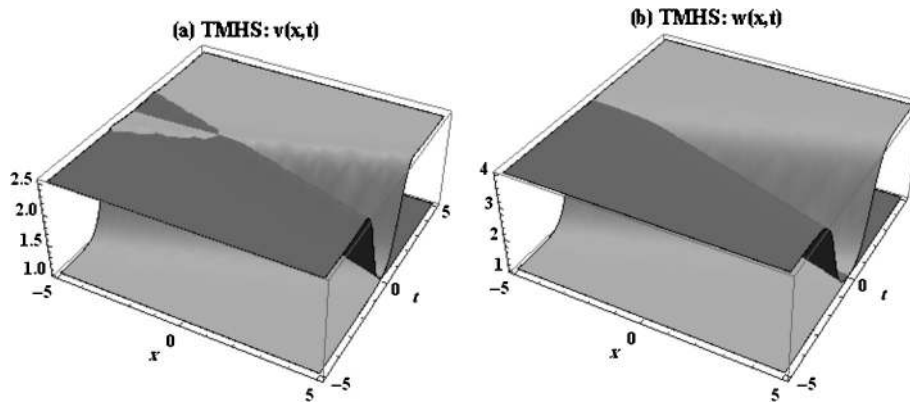


Fig. 4. Overlapping of two-shock waves obtained for TMHS given in (3.35) by means of Kudryashov-expansion method. (a) $v(x, t)$: $a_0 = 1, k = -1, \mu = 1, d = 0.5, s = 3$. (b) $w(x, t)$: $b_0 = 1, k = -1, \mu = 1, d = 0.5, s = 3$.

where Z satisfies the same differential equation given in (2.18). Now, we differentiate both v and w all the necessary order derivatives and insert them into the system (3.25). Then, we set the coefficients of same power Z^i to zero to get a nonlinear algebraic system in the unknowns $a_0, a_1, a_2, b_0, b_1, b_2, \mu$ and λ . The resulting system is insolvable unless we add a reasonable condition which is

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \pm 1. \tag{3.34}$$

Consequently, one more new solution of kink-type is obtained and given by

$$v(x, t) = a_0 + \frac{7\mu^2}{2(1 + d e^{\mu(x \pm st)})} - \frac{2\mu^2}{(1 + d e^{\mu(x \pm st)})^2},$$

$$w(x, t) = b_0 + \frac{7\mu^2}{1 + d e^{\mu(x \pm st)}} + \frac{\mu^2}{(1 + d e^{\mu(x \pm st)})^2}, \tag{3.35}$$

where $a_0, b_0, d \neq 0$ and $\mu \neq 0$ are free parameters. Fig. 4, represents the physical shapes of the obtained solution depicted in (3.35) to the system TMHS.

4. Discussion

We successfully introduced two new mathematical models created by using the definition of Korsunsky. One is a single equation “TMKKBK”, and the other is a system of equations “TMHS”. Both were introduced for the first time in this work. Both models belong to a new set of nonlinear equations that describes the motion of two-wave propagating in the same direction simultaneously and overlapping without any distortion of their physical shape and for this reason it is being called two-mode equations.

Our goal was to identify the physics and dynamics of these new types of equations by exploring some of their solitary-wave solutions. To achieve this, we used an adaptation of some methods being implemented in literature. Since there are similar methods in terms of its constructions and we are not taken into account to show the advantage of one method over another. We used two different schemes, one which gives shock-wave or soliton-wave and other gives periodic waves.

We believe that such new equations supported by our findings will play an important role in the field of wave carriers. For instance, the obtained two-waves can be regarded as a barrier waves to increase the strength of transmitting different signals-data. Further, these two-waves can be used as a divider of big data to decrease the load on single routers.

5. Conclusion

The current work presented two new equations; the two-mode KdV–Burgers–Kuramoto equation and two-mode Hirota–Satsuma system.

These types of equations describe the propagation of two-directional-waves moving with distributed phase velocity and nonlinearity-dispersion parameters. Two effective schemes are used to extract possible solutions to both models. Different types of solutions are reported where some labeled are remarkable being a combination of kink and soliton waves. 3D and 2D plots are provided to address two issues; the physical shape of such two-mode equations and the effect of increasing the phase velocity on the overlapping phase-angle between the spreading of the depicted two-waves.

As future work, we aim to study time-fractional version of two-mode equations and use different new approaches [33, 34] to extract analytical solutions. Another interesting aspect worth to be investigated if we consider two-mode equation with time-delay acting on the dispersion terms [35, 36, 37]. Finally, we aim to explore Lump soliton solutions [38, 39, 40] for such two-mode equations.

Declarations

Author contribution statement

M. Alquran, I. Jaradat: Conceived and designed the experiments; Analyzed and interpreted the data.

M. Ali, N. Al-Ali: Contributed reagents, materials, analysis tools or data; Wrote the paper.

S. Momani: Performed the experiments; Analyzed and interpreted the data.

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Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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