Applied Mathematics Letters 25 (2012) 1590–1595

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Developing an algorithm for triad design

Tareq M. Abu Saa [a,*], Haslinda Ibrahim [b], El Mostafa Kalmoun [b]

[a] Ajman University of Science and Technology Network, Ajman, United Arab Emirates

[b] College of Arts and Sciences, Universiti Utara Malaysia, 6010 Sintok, Kedah, Malaysia

ARTICLE INFO

Article history:
Received 27 February 2011
Received in revised form 10 January 2012
Accepted 21 January 2012

Keywords:
Compatible factorization
Triad design
Starter

ABSTRACT

The aim of this paper is to construct a method for developing a triad design on v objects $TD(v)$, for the general case $v = 6n + 1$. This method depends on analyzing the triples to construct a starter for $TD(v)$, denoted by $STD(v)$, using interval techniques of the triples in the starter. We illustrate this construction by considering the cases $v = 13$ and $v = 19$ towards providing the design for the general case.

1. Introduction

A compatible factorization of a graph of order v, denoted by $CF(v)$, is a $v \times \frac{v-1}{2}$ array that satisfies the following conditions [1–3].

(i) The entries in row m form a near-1 factor with focus m.
(ii) The triples associated with the rows contain no repetitions.

A triad design on v objects, $TD(v)$, is a method for listing the distinct triples of $\binom{v}{3}$. It is a triple system formed from compatible factorization $CF(v)$ [4,5]. It is known that $TD(v)$ exists iff $v = 1$ or $5 \pmod 6$ [5]. For example, if $v = 7$, that is, a set of seven points labeled 1, 2, 3, 4, 5, 6, 7, then the compatible factorization $CF(7)$ is as shown in Table 1. On appending C_1 with C_2, C_1 with C_3, and C_1 with C_4, we obtain 21 distinct triples in $CF(7)$. Moreover, $TD(7) = CF(7) \cup \overline{CF(7)}$, where $\overline{CF(7)}$ is the completion of $CF(7)$. Clearly, $TD(7)$ consists of $\binom{7}{3} = 35$distinct triples.

Table 1 shows that repeated addition of 1 modulo 7 to the first row, called starter, produces all the distinct triples. Generally, the starter of $TD(v)$, $STD(v)$, is the set of triples that generates all the triples in the design by repeated addition of 1 modulo v. Once we construct the starter, listing all distinct triples can be done by repeated addition of 1 modulo v. Our aim in this paper is to present a new method to construct the starter. This method depends on analyzing the triples using interval techniques of the triples in the starter. We illustrate this construction by considering the cases $v = 13$ and $v = 19$ towards deducing the general case of $v = 6n + 1$.

2. Properties of $STD(v)$

It is easy to conclude from the above discussion that $STD(v) = SCF(v) \cup \overline{SCF(v)}$. The following lemma provides the number of triples in $STD(v)$, denoted by $|STD(v)|$.

* Corresponding author. Tel.: +971 507300275.
E-mail address: tareqabusaa@yahoo.com (T.M. Abu Saa).

0893-9659/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2012.01.017

Table 1
TD(7).

$CF(7)$				$\overline{CF(7)}$	
C_1	C_2	C_3	C_4		
1	2 7	3 6	4 5	2 3 5	7 6 4
2	3 1	4 7	5 6	3 4 6	1 7 5
3	4 2	5 1	6 7	4 5 7	2 1 6
4	5 3	6 2	7 1	5 6 1	3 2 7
5	6 4	7 3	1 2	6 7 2	4 3 1
6	7 5	1 4	2 3	7 1 3	5 4 2
7	1 6	2 5	3 4	1 2 4	6 5 3

Table 2
$S_rTD(13)$, $1 \le r \le 3$.

k	**1**	2	…	**6**	7	8	…	13	**14**	15	16	17	**18**	19	20	21	22
$S_1TD(13)$	1	1	…	1	13	2	…	13	2	12	3	12	3	11	4	10	5
$S_2TD(13)$	2	3	…	7	12	3	…	9	6	11	4	10	5	10	5	9	6
$S_3TD(13)$	13	12		8	4	11	…	7	8	6	9	7	8	7	8	7	8

Table 3
Intervals of k and $[S_rTD(13)]_k$, $1 \le r \le 3$.

Intervals of k	$[S_1TD(13)]_k$	$[S_2TD(13)]_k$	$[S_3TD(13)]_k$
$1 \le k \le 6$	1, 1, 1, 1, 1, 1	2, 3, 4, 5, 6, 7	13, 11, 12, 10, 9, 8
$7 \le k \le 14$	13, 2, 13, 2, 13, 2, 13, 2	12, 3, 11, 4, 10, 5, 9, 6	4, 11, 5, 10, 6, 9, 7, 8
$15 \le k \le 18$	12, 3, 12, 3	11, 4, 10, 5	6, 9, 7, 8
$19 \le k \le 22$	11, 4, 10, 5	10, 5, 9, 6	7, 8, 7, 8

Lemma 2.1. *If $v = 6n + 1$, then $|STD(v)|$ is equal to $n(6n - 1)$.*

Proof. The number of triples of $TD(v)$ is equal to $|TD(v)| = \binom{v}{3} = \binom{6n+1}{3} = \frac{(6n+1)(6n)(6n-1)}{6} = n(6n + 1)(6n - 1)$. By the definition of $CF(v)$, the number of rows of $TD(v)$ is equal to $v = 6n + 1$. Hence, $|STD(v)| = n(6n - 1)$. $\square$

Each triple in $STD(v)$ consists of three elements (numbers, objects). For $1 \le r \le 3$, we denote by $S_rTD(v)$ the r-th numbers in each triple and call them the r-th elements of $STD(v)$.

Example 2.1. From Table 1, $STD(7) = SCF(7) \cup \overline{SCF(7)} = \{1\ 2\ 7;\ 1\ 3\ 6;\ 1\ 4\ 5;\ 2\ 3\ 5;\ 7\ 6\ 4\}$. Therefore $S_1TD(7)$: 1, 1, 1, 2, 7 ; $S_2TD(7)$: 2, 3, 4, 3, 6 and $S_3TD(7)$: 7, 6, 5, 5, 4.

3. Interval constructions of *STD*(13)

The algorithm for constructing $STD(13)$ can be found in [6,4,7]. We have

$$\begin{aligned} STD(13) &= SCF(13) \cup \overline{SCF(13)} = \{1\ 2\ 13;\ 1\ 3\ 12;\ 1\ 4\ 11,\ 1\ 5\ 10;\ 1\ 6\ 9;\ 1\ 7\ 8\} \\ &\quad \cup \{13\ 12\ 4;\ 2\ 3\ 11;\ 13\ 11\ 5;\ 2\ 4\ 10;\ 13\ 10\ 6;\ 2\ 5\ 9;\ 13\ 9\ 7;\ 2\ 6\ 8;\ 12\ 11\ 6; \\ &\qquad 3\ 4\ 9;\ 12\ 10\ 7;\ 3\ 5\ 8;\ 11\ 10\ 7;\ 4\ 5\ 8;\ 10\ 9\ 7;\ 5\ 6\ 8\}. \end{aligned}$$

Our aim in this section is to develop a new method to construct $STD(13)$. Let k be the number of triples in $STD(13)$. By Lemma 2.1, $1 \le k \le 22$. Let $[S_rTD(13)]_k$ be the k-th element in $S_rTD(13)$ for $1 \le r \le 3$. We are able to summarize $STD(13)$ in terms of $S_rTD(13)$, as shown in Table 2.

Clearly, k, the number of triples in $STD(13)$, can be divided into four intervals (periods). These intervals and the corresponding elements in $S_1TD(13)$, $S_2TD(13)$ and $S_3TD(13)$ are given in Table 3.

We observe that the corresponding elements in $S_rTD(13)$, $1 \le r \le 3$, need a formula to produce them. Let f denotes the first number of the interval. The formula $y = \frac{1}{2}[k - f + \mathrm{mod}(\frac{n+k}{2}) + 1]$ can be used to produce the corresponding elements in $S_rTD(13)$. From Tables 2 and 3, the construction of $S_rTD(13)$ is as follows.

$$[S_1TD(13)]_k = \begin{cases} 1 & & 1 \le k \le 6 \\ 2 & & 7 \le k \le 14,\ k \text{ is even} \\ 13 & & 7 \le k \le 14,\ k \text{ is odd} \\ 3 & \text{if} & 15 \le k \le 18,\ k \text{ is even} \\ 12 & & 15 \le k \le 18,\ k \text{ is odd} \\ 3 + y & & 19 \le k \le 22,\ k \text{ is even} \\ 12 - y & & 19 \le k \le 22,\ k \text{ is odd} \end{cases}$$

$$[S_2TD(13)]_k = \begin{cases} 1+k & 1 \le k \le 6 \\ 2+y & 7 \le k \le 14,\ k \text{ is even} \\ 13-y & 7 \le k \le 14,\ k \text{ is odd} \\ 3+y \quad \text{if} & 15 \le k \le 18,\ k \text{ is even} \\ 12-y & 15 \le k \le 18,\ k \text{ is odd} \\ 4+y & 19 \le k \le 22,\ k \text{ is even} \\ 11-y & 19 \le k \le 22,\ k \text{ is odd} \end{cases} \qquad [S_3TD(13)]_k = \begin{cases} 14-k & 1 \le k \le 6 \\ 12-y & 7 \le k \le 14,\ k \text{ is even} \\ 3+y & 7 \le k \le 14,\ k \text{ is odd} \\ 11-y \quad \text{if} & 15 \le k \le 18,\ k \text{ is even} \\ 4+y & 15 \le k \le 18,\ k \text{ is odd} \\ 8 & 19 \le k \le 22,\ k \text{ is even} \\ 7 & 19 \le k \le 22,\ k \text{ is odd.} \end{cases}$$

Now, we can use the starter $STD(13)$ and repeated addition of 1 modulo 13 to list $TD(13)$, as shown in Table 4.

4. Interval constructions of *STD*(19)

Similarly, we can construct $STD(19)$ in terms of $S_rTD(19)$, $1 \le r \le 3$, as shown in Table 5.
Using the same notation as in Section 3, the constructions of $S_rTD(19)$ for $1 \le r \le 3$ are as follows:

$$[S_1TD(19)]_k = \begin{cases} 1 & 1 \le k \le 9 \\ 2 & 10 \le k \le 23,\ k \text{ is even} \\ 19 & 10 \le k \le 23,\ k \text{ is odd} \\ 3 & 24 \le k \le 33,\ k \text{ is even} \\ 18 & 24 \le k \le 33,\ k \text{ is odd} \\ 4 & 34 \le k \le 39,\ k \text{ is even} \\ 17 \quad \text{if} & 34 \le k \le 39,\ k \text{ is odd} \\ 5 & 40 \le k \le 43,\ k \text{ is even} \\ 16 & 40 \le k \le 43,\ k \text{ is odd} \\ 6 & 44 \le k \le 47,\ k \text{ is even} \\ 15 & 44 \le k \le 47,\ k \text{ is odd} \\ 6+y & 48 \le k \le 51,\ k \text{ is even} \\ 15-y & 48 \le k \le 51,\ k \text{ is odd} \end{cases}$$

$$[S_2TD(19)]_k = \begin{cases} 1+k & 1 \le k \le 9 \\ 2+y & 10 \le k \le 23,\ k \text{ is even} \\ 19-y & 10 \le k \le 23,\ k \text{ is odd} \\ 3+y & 24 \le k \le 33,\ k \text{ is even} \\ 18-y & 24 \le k \le 33,\ k \text{ is odd} \\ 4+y & 34 \le k \le 39,\ k \text{ is even} \\ 17-y \quad \text{if} & 34 \le k \le 39,\ k \text{ is odd} \\ 5+y & 40 \le k \le 43,\ k \text{ is even} \\ 16-y & 40 \le k \le 43,\ k \text{ is odd} \\ 6+y & 44 \le k \le 47,\ k \text{ is even} \\ 15-y & 44 \le k \le 47,\ k \text{ is odd} \\ 7+y & 48 \le k \le 51,\ k \text{ is even} \\ 14-y & 48 \le k \le 51,\ k \text{ is odd} \end{cases} \qquad [S_3TD(19)]_k = \begin{cases} 20-k & 1 \le k \le 9 \\ 18-y & 10 \le k \le 23,\ k \text{ is even} \\ 3+y & 10 \le k \le 23,\ k \text{ is odd} \\ 16-y & 24 \le k \le 33,\ k \text{ is even} \\ 5+y & 24 \le k \le 33,\ k \text{ is odd} \\ 14-y & 34 \le k \le 39,\ k \text{ is even} \\ 7+y \quad \text{if} & 34 \le k \le 39,\ k \text{ is odd} \\ 13-y & 40 \le k \le 43,\ k \text{ is even} \\ 8+y & 40 \le k \le 43,\ k \text{ is odd} \\ 13-y & 44 \le k \le 47,\ k \text{ is even} \\ 8+y & 44 \le k \le 47,\ k \text{ is odd} \\ 11 & 48 \le k \le 51,\ k \text{ is even} \\ 10 & 48 \le k \le 51,\ k \text{ is odd.} \end{cases}$$

5. Interval constructions for the general case *STD*(6*n* + 1)

Form the cases $v = 7,\ 13,\ 19$, we are able to establish the table of intervals as a result of analyzing the triples in the designs of the previous cases; see Table 6.

From Table 6, we deduce the following results.

Remark 5.1. The number of intervals of $STD(6n+1)$ is equal to $3n-2$.

The general rules of the intervals of the general case $v = 6n+1$ are as follows.

(1) The first interval is $1 \le k \le 3n$.
(2) Other intervals except the last one are generated as follows. Let i denote the interval number starting from the second interval until the second to last one. It can be observed from Remark 5.1 that the values of i are $1 \le i \le 3n-4$. The last numbers of the intervals, denoted by l, are constructed as follows:

$$l = 9n - 4 = (6(1)+3)\,n - (2(1)^2 + 2(1)).$$
$$l = 15n - 12 = (6(2)+3)\,n - (2(2)^2 + 2(2)).$$
$$l = 21n - 24 = (6(3)+3)\,n - (2(3)^2 + 2(3)).$$
$$l = (6i+3)\,n - (2i^2 + 2i), \quad \text{where } 1 \le i \le n.$$

Table 4
TD(13).

1	2	13	1	3	12	1	4	11	…	10	9	7	5	6	8
2	3	1	2	4	13	2	5	12	…	11	10	8	6	7	9
3	4	2	3	5	1	3	6	13	…	12	11	9	7	8	10
4	5	3	4	6	2	4	7	1	…	13	12	10	8	9	11
5	6	4	5	7	3	5	8	2	…	1	13	11	9	10	12
6	7	5	6	8	4	6	9	3	…	2	1	12	10	11	13
7	8	6	7	9	5	7	10	4	…	3	2	13	11	12	1
8	9	7	8	10	6	8	11	5	…	4	3	1	12	13	2
9	10	8	9	11	7	9	12	6	…	5	4	2	13	1	3
10	11	9	10	12	8	10	13	7	…	6	5	3	1	2	4
11	12	10	11	13	9	11	1	8	…	7	6	4	2	3	5
12	13	11	12	1	10	12	2	9	…	8	7	5	3	4	6
13	1	12	13	2	11	13	3	10	…	9	8	6	4	5	7

Table 5
$S_rTD(19)$, $1 \leq r \leq 3$.

k	**1**	2	…	**9**	10	11	…	22	**23**	24	25	…	32	**33**	34	35	…	38	**39**	40	41	42	**43**	44	45	46	**47**	48	49	50	**51**
$S_1TD(19)$	1	1	1	1	2	19	…	2	19	3	18	…	3	18	4	17	…	4	17	5	16	5	16	6	15	6	15	7	14	8	13
$S_2TD(19)$	2	3	…	10	3	18	…	9	12	4	17	…	8	13	5	16	…	7	14	6	15	7	14	7	14	8	13	8	13	9	12
$S_3TD(19)$	19	18	…	11	17	4	…	11	10	15	6	…	11	10	13	8	…	11	10	12	9	11	10	12	9	11	10	11	10	11	10

Table 6
Intervals of $STD(6n+1)$.

Intervals of k	$v = 7,\ n = 1$	$v = 13,\ n = 2$	$v = 19,\ n = 3$	…	$v = 6n + 1$
	$1 \leq k \leq 3$	$1 \leq k \leq 6$	$1 \leq k \leq 9$	…	$1 \leq k \leq 3n$
		$7 \leq k \leq 14$	$10 \leq k \leq 23$	…	$3n + 1 \leq k \leq 9n - 4$
		$15 \leq k \leq 18$	$24 \leq k \leq 33$	…	$9n-3 \leq k \leq 15n-12$
		$19 \leq k \leq 22$	$34 \leq k \leq 39$	…	$15n - 11 \leq k \leq 21n - 24$
			$40 \leq k \leq 43$	…	$21n - 23 \leq k \leq 27n - 40$
			$44 \leq k \leq 47$	…	$27n - 39 \leq k \leq 33n - 58$
			$48 \leq k \leq 51$	…	$33n - 57 \leq k \leq 39n - 76$
No. of intervals	1	4	7		$3n - 2$

If $1 + n \leq i \leq 3n - 4$, then a value denoted by a_t, defined below, must be added to obtain the desired last numbers of the intervals. Hence, the last number of the intervals in this case is equal to $l + a_t$. If $i = n + t$, then clearly $1 \leq t \leq 2n - 4$. We define $a_t = a_{t-2} + 6t - 4$, $a_0 = 0$ and $a_1 = 2$.

Similarly, the first number f of the intervals is equal to

$$f = [6(i-1) + 3]n - [2(i-1)^2 + 2(i-1)] + 1, \quad \text{where } 1 \leq i \leq n.$$

Also, if $1+n \leq i \leq 3n-4$, then a value denoted by a_{t-1} must be added to obtain the desired lower numbers of the intervals. Therefore, the first number of the general intervals in this case is equal to $f + a_{t-1}$, where $i = n + t$ and $1 \leq t \leq 2n - 4$.

The above discussion and notation are summarized in the following interval generalization concerning the patterns of the intervals for the general case $v = 6n + 1$.

Interval generalization. Using the same notation as above, the patterns of the intervals for the general case $v = 6n+1$, where $n > 1$, are the following.

1. The first interval is $1 \leq k \leq 3n$.
2. Other intervals except the last one are explained as follows:

$$f \leq k \leq l, \quad \text{where } 1 \leq i \leq n, \quad \text{and} \quad f + a_{t-1} \leq k \leq l + a_t, \quad \text{where } 1 + n \leq i \leq 3n - 4,\ i = n + t,\ 1 \leq t \leq 2n - 4.$$

   Also $a_0 = 0$, $a_1 = 2$, and, if $t \geq 2$, then $a_t = a_{(t-2)} + (6t - 4)$.
3. The last interval is $l + a_t + 1 \leq k \leq n(6n - -1)$, $t = 2n - 4$.

Example 5.1. Let $n = 3$; that is, $v = 19$. By Remark 5.1, the number of intervals is equal to 7. Clearly, $1 \leq i \leq 5$ and $1 \leq t \leq 2$. We apply interval generalization to obtain the intervals for the case $v = 19$.

1. The first interval is $1 \le k \le 3n$; that is, $1 \le k \le 9$.
2. If $1 \le i \le 3$, then the second, third, and fourth intervals can be easily calculated using $f \le k \le l$; that is, $[6(i-1)+3](3) - [2(i-1)^2 + 2(i-1)] + 1 \le k \le (6i+3)(3) - (2i^2+2i)$, by substituting $i = 1, 2$, and 3. Thus, $10 \le k \le 23$, $10 \le k \le 23$ and $34 \le k \le 39$ are the desired intervals.

   Now, if $4 \le i \le 5$, then the fifth and sixth intervals can be easily calculated using $f + a_{t-1} \le k \le l + a_t$, or $[6(i-1)+3](3) - [2(i-1)^2 + 2(i-1)] + 1 + a_{t-1} \le k \le (6i+3)(3) - (2i^2+2i) + a_t$, by substituting $i = 4$ and $i = 5$. Thus, $40 \le k \le 43$, $44 \le k \le 47$ are the desired intervals.
3. The last interval is $l + a_t + 1 \le k \le n(6n - -1)$, where $t = 2n - 4$, that is, $48 \le k \le 51$.

We are now in a position to give the construction of $S_rTD(v)$ for $1 \le r \le 3$. Let $[S_rTD(v)]_k$ be the k-th element in $S_rTD(v)$ for $1 \le r \le 3$. Let $1 \le i \le 3n - 4$ denote the number of intervals of $STD(v)$ except the first and the last interval. When $i > n$, we set $t = i - n$. Clearly, $1 \le t \le 2n - 4$. Define a_t as above. In addition to the above notation, we use the formula $y = \frac{1}{2}[k - f + \mathrm{mod}(\frac{n+k}{2}) + 1]$.

Theorem 5.2. *Following the above notation, the constructions of $S_rTD(v)$ for $1 \le r \le 3$ are as follows:*

$$[S_1TD(v)]_k = \begin{cases} 1 & & 1 \le k \le 3n & \text{(a)} \\ i+1 & & f \le k \le l,\ 1 \le i \le n,\ k \text{ is even} & \text{(b)} \\ 6n+2-i & & f \le k \le l,\ 1 \le i \le n,\ k \text{ is odd} & \\ i+1 & \text{if} & f + a_{t-1} \le k \le l + a_t,\ i = n+t,\ k \text{ is even} & \text{(c)} \\ 6n+2-i & & f + a_{t-1} \le k \le l + a_t,\ i = n+t,\ k \text{ is odd} & \\ & & n+1 \le i \le 3n-4,\ 1 \le t \le 2n-4 & \text{(d)} \\ 3n-3+y & & l + a_t + 1 \le k \le n(6n-1),\ k \text{ is even},\ t = 2n-4 & \text{(e)} \\ 3n+6-y & & l + a_t + 1 \le k \le n(6n-1),\ k \text{ is odd},\ t = 2n-4 & \text{(f)} \end{cases}$$

$$[S_2TD(v)]_k = \begin{cases} k+1 & & 1 \le k \le 3n & \text{(a)} \\ i+1+y & & f \le k \le l,\ 1 \le i \le n,\ k \text{ is even} & \text{(b)} \\ 6n+2-i-y & & f \le k \le l,\ 1 \le i \le n,\ k \text{ is odd} & \\ i+1+y & \text{if} & f + a_{t-1} \le k \le l + a_t,\ i = n+t,\ k \text{ is even} & \text{(c)} \\ 6n+2-i-y & & f + a_{t-1} \le k \le l + a_t,\ i = n+t,\ k \text{ is odd} & \\ & & n+1 \le i \le 3n-4,\ 1 \le t \le 2n-4 & \text{(d)} \\ 3n-3+y & & l + a_t + 1 \le k \le n(6n-1),\ k \text{ is even},\ t = 2n-4 & \text{(e)} \\ 3n+5-y & & l + a_t + 1 \le k \le n(6n-1),\ k \text{ is odd},\ t = 2n-4 & \text{(f)} \end{cases}$$

$$[S_3TD(v)]_k = \begin{cases} 6n+2-k & & 1 \le k \le 3n & \text{(a)} \\ 6n+2-2i-y & & f \le k \le l,\ 1 \le i \le n,\ k \text{ is even} & \text{(b)} \\ 2i+1+y & & f \le k \le l,\ 1 \le i \le n,\ k \text{ is odd} & \\ 6n+2-2i-y+C_j & \text{if} & f + a_{t-1} \le k \le l + a_t,\ i = n+t,\ k \text{ is even} & \text{(c)} \\ 2i+1+y-C_j & & f + a_{t-1} \le k \le l + a_t,\ i = n+t,\ k \text{ is odd} & \\ & & n+1 \le i \le 3n-4,\ 1 \le t \le 2n-4 & \text{(d)} \\ 3n+2 & & l + a_t + 1 \le k \le n(6n-1),\ k \text{ is even},\ t = 2n-4 & \text{(e)} \\ 3n+1 & & l + a_t + 1 \le k \le n(6n-1),\ k \text{ is odd},\ t = 2n-4 & \text{(f)}, \end{cases}$$

where $C_j = C_{j-1} + 1 + \mathrm{mod}(\frac{j+1}{2}),\ j = i - n,\ j < 1,\ \ C_j = 0.$

Proof. We prove the construction related to $S_1TD(v)$. The proofs of the other two are similar.

(a) If $1 \le k \le 3n$, then $S_1TD(v)$ is the first element of $FC(v)$, which is always equal 1.
(b) From the construction of $S_1TD(13)$ and $S_1TD(19)$, we can see that, if $f \le k \le l$ is even and $1 \le i \le n$, then $S_1TD(13)$: 2, 3, and $S_1TD(19)$: 2, 3, 4; that is, $i + 1$. Also,if k is odd, then $S_1TD(13)$: 13, 12 and $S_1TD(19)$: 19, 18, 17; that is, $v + 1 - i = 6n + 2 - -i$.
(c) The same applies for $f + a_{t-1} \le k \le l + a_t$, $n + 1 \le i \le 3n - 4$, $i = n + t$, and $1 \le t \le 2n - 4$.
(d) Clearly, if $i = n + t$ and $n + 1 \le i \le 3n - 4$, then $1 \le i - n \le 2n - 4$; that is, $1 \le t \le 2n - 4$.
(e) For the last interval $l + a_{t+1} \le k \le n(6n - 1)$, $t = 2n - 4$, we see that $S_1TD(13)$ is equal to 4, 5 and $S_1TD(19)$ is equal to 7, 8 when k is even. Therefore, it is easy to deduce that $[S_1TD(v)]_k = 3n - 3 + y$, where y is the rule defined above.
(f) Finally, when k is odd in the last interval, then $S_1TD(13)$ is equal to 11, 10 and $S_1TD(19)$ is equal to 14, 13. Therefore, $[S_1TD(v)]_k = 3n - 6 - y$. □

6. Conclusion

We have constructed a method for developing a triad design on v objects, $TD(v)$, for the general case $v = 6n+1$. A similar method can be developed for the case $v = 6n+5$, based on analyzing the triples in the starter of $TD(6n+5)$. Moreover, one can ask if a near-triad design (*NTD*) can be constructed for the cases $v = 6n$, $v = 6n+2$, $v = 6n+3$, or $v = 6n+4$, since it is impossible to construct a *TD* for them. We expect that *coverings* and *packings* can be used to construct an *NTD* [8,9].

References

[1] H. Ibrahim, Compatible factorizations and three-fold triple systems, Bulletin of the Malaysian Mathematical Sciences Society 29 (2005) 125–130.
[2] H. Ibrahim, W.D. Wallis, An enumeration of triad designs, Journal of Combinatorics Information and System Sciences 30 (2005) 5–17.
[3] W.D. Wallis, One-Factorization, Kluwer, Boston, 1997.
[4] T.M. Abu Saa, H. Ibrahim, Z. Omer, R. Rahim, On particular enumeration for *TD*(13), in: Proceeding of the 5th Social Economic & Information Technology (SEit), Hatyai, Thailand, 2010, pp. 133.
[5] H. Ibrahim, A Family of Designs for Triads and Related Factorizations, Ph.D. Thesis, Southern Illinois University, Carbondale. 2003.
[6] T.M. Abu Saa, H. Ibrahim, A new algorithm for some triad designs, in: The First International Conference on Mathematics and Statistics. Proceeding, American University of Sharjah, Sharjah, United Arab Emirates, 2010, pp. 176.
[7] H. Ibrahim, T.M. Abu Saa, Z. Omar, R. Rahim, Difference set method and its application in developing some triad designs, in: Proceedings of the 5th Asian Mathematical Conference, Malaysia, 2009.
[8] C.C. Lindner, C.A. Rodger, Design Theory, CRC Press, New York, 1997.
[9] W.D. Wallis, Combinatorial Designs, Marcel Dekker, New York, 1997.