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Control and switching synchronization of fractional order chaotic systems using active control technique



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ABSTRACT

This paper discusses the continuous effect of the fractional order parameter of the Lü system where the system response starts stable, passing by chaotic behavior then reaching periodic response as the fractional-order increases. In addition, this paper presents the concept of synchronization of different fractional order chaotic systems using active control technique. Four different synchronization cases are introduced based on the switching parameters. Also, the static and dynamic synchronizations can be obtained when the switching parameters are functions of time. The nonstandard finite difference method is used for the numerical solution of the fractional order master and slave systems. Many numeric simulations are presented to validate the concept for different fractional order parameters.

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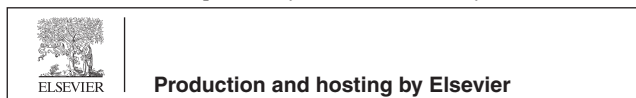
Introduction

During the last few decades, fractional calculus has become a powerful tool in describing the dynamics of complex systems which appear frequently in several branches of science and engineering. Therefore fractional differential equations and

their numerical techniques find numerous applications in the field of viscoelasticity, robotics, feedback amplifiers, electrical circuits, control theory, electro analytical chemistry, fractional multi-poles, chemistry and biological sciences [1–12].

The chaotic dynamics of fractional order systems began to attract a great deal of attention in recent years due to the ease of their electronic implementations as discussed before [13,14]. Due to the very high sensitivity of these chaotic systems which is required for many applications, there was a need to discuss the coupling of two or more dissipative chaotic systems which is known as synchronization. Chaotic synchronization has been applied in many different fields, such as biological and physical systems, structural engineering, ecological models [15,16].

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Pecora and Carroll [15] were the first to introduce the concept of synchronization of two systems with different initial conditions. Many chaotic synchronization schemes have also been introduced during the last decade such as adaptive control, time delay feedback approach [17,18], nonlinear feedback synchronization, and active control [19]. However, most of these methods have been tested for two identical chaotic systems. When Ho and Hung [19] presented and applied the concept of active control method on the synchronization of chaotic systems, many recent papers investigated this technique for different systems and in different applications [20,21]. The synchronization of three chaotic fractional order Lorenz systems with bidirectional coupling in addition to the chaos synchronization of two identical systems via linear control was investigated [22,23]. Moreover, two different fractional order chaotic systems can be synchronized using active control [24]. The hyper-chaotic synchronization of the fractional order Rössler system which exists when its order is as low as 3.8 was shown by Yua and Lib [25]. Recently the consistency for the improvement of models based on fractional order differential structure has increased in the research of dynamical systems [26]. In addition, many researchers have studied the control of systems in different applications [27,28], in addition to the circuit and electromagnetic theories as shown by others [3,4,10–12,29].

Several analytical and numerical methods have been proposed to solve the fractional order differential equations for example the nonstandard finite difference schemes (NSFDs), developed by Mickens [30,31] have shown great potential in recent applications [32,33].

There are two aims for this paper, the first aim is to study the proper fractional order range which exhibits chaotic behavior for the Lü system. More than thirty cases are investigated for different orders and changing only a single system parameter. Stable, periodic and chaotic responses are shown for each system parameter but with different fractional order ranges. The second aim is to discuss the active technique for the synchronization of two different fractional order chaotic systems and using two on/off switches. Based on the proposed technique, static and dynamic synchronization can be obtained in four different cases. The numerical solutions of the fractional order for the master, slave and error systems are computed using NSFD.

In ‘Fundamentals of fractional order’ the basic fundamentals of the fractional order will be discussed. ‘Grünwald–Letnikov approximation’ will introduce the effect of the fractional order parameter of the fractional Lü system on the output response. The concept of active control using two on/off switches for the synchronization between two different chaotic systems will be proposed in ‘Non-standard Discretization’. Four different static and dynamic synchronization cases will be introduced in ‘Effect of the fractional order parameter on the Lü system response’ based on changing the switching parameters with time. Finally, conclusions are drawn in the last section.

Fundamentals of fractional order

Although the concept of the fractional calculus was discussed in the same time interval of integer order calculus, the complexity and the lack of applications postponed its progress till

a few decades ago. Recently, most of the dynamical systems based on the integer-order calculus have been modified into the fractional order domain due to the extra degrees of freedom and the flexibility which can be used to precisely fit the experimental data much better than the integer-order modeling. For example, new fundamentals have been investigated in the fractional order domain for the first time and do not exist in the integer-order systems such as those presented in [4,6,9–12]. The Caputo fractional derivative of order α of a continuous function $f: R^+ \rightarrow R$ is defined as follows:

$$D^\alpha f(t) \equiv \frac{d^\alpha f(t)}{dt^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t) & \alpha = m \end{cases} \quad (1)$$

where m is the first integer greater than α , and $\Gamma(\cdot)$ is the Gamma function and is defined by:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \Gamma(z+1) = z\Gamma(z) \quad (2)$$

In this section, some basic definitions and properties of the fractional calculus theory and nonstandard discretization are discussed.

Grünwald–Letnikov approximation

The Grünwald–Letnikov method of approximation for the one-dimensional fractional derivative is as follows [34]:

$$D^\alpha x(t) = f(t, x) \quad (3)$$

$$D^\alpha x(t) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{j=0}^{t/h} (-1)^j \binom{\alpha}{j} x(t-jh) \quad (4)$$

where $\alpha > 0$, D^α denotes the fractional derivative. $N = [t/h]$, and h is the step size. Therefore, Eq. (3) is discretized as follows:

$$\sum_{j=0}^{n+1} c_j^\alpha x(t-jh) = f(t_n, x(t_n)), \quad n = 1, 2, 3, \dots, \quad (5)$$

where $t_n = nh$ and c_j^α are the Grünwald–Letnikov coefficients defined as:

$$c_j^\alpha = \left(1 - \frac{1+\alpha}{j}\right) c_{j-1}^\alpha, \quad \text{and} \quad c_0^\alpha = h^{-\alpha}, \quad j = 1, 2, 3, \dots \quad (6)$$

Nonstandard discretization

The nonstandard discretization technique is a general scheme where we replace the step size h by a function $\varphi(h)$. By applying this technique and using the Grünwald–Letnikov discretization method, it yields the following relations

$$x_{n+1} = \frac{-\sum_{j=1}^{n+1} c_j^\alpha x_{n+1-j} + f_1(t_{n+1}, x_{n+1})}{c_0^\alpha} \quad (7)$$

where $c_0^\alpha = (\varphi_1(h))^{-1}$ are functions of the step size $h = \Delta t$, with the following properties:

$$\varphi_1(h) = h + O(h^2), \quad \text{where} \quad h \rightarrow 0 \quad (8)$$

Examples of the function $\varphi_1(h)$ that satisfies (8) is $h, \sin(h), \sinh(h), e^h - 1$, and in most applications, the general choice of $\varphi_1(h)$ is $(1 - e^{-R_1 h})/R_1$, where the function R_1 can be chosen as

$$R_1 = \max \left(\left| \frac{\partial f_1}{\partial x} \right| \right) \quad (9)$$

The multiplication terms can be replaced by nonlocal discrete representations. For example,

$$y^2 \approx y_k y_{k+1}, \quad xy \approx 2x_{n+1}y_n - x_{n+1}y_{n+1} \quad (10)$$

Effect of the fractional order parameter on the Lü system response

The Fractional order Lü system is the lowest-order chaotic system amongst all of chaotic systems [35]. The minimum effective dimension reported is 0.30. The system is given by

$$\begin{aligned} D^\alpha x(t) &= a(y(t) - x(t)) \\ D^\alpha y(t) &= by(t) - x(t)y(t) \\ D^\alpha z(t) &= x(t)y(t) - cz(t) \end{aligned} \quad (11)$$

where a, b , and c are the system parameters, (x, y, z) are the state variables, and α is the fractional order. Now, we apply the NSFD to obtain the numerical solution for the fractional

order Lü system. Using the Grünwald–Letnikov discretization method and applying the NSFD scheme by replacing the step size h by a function $\varphi(h)$ and applying this form in (7) for the nonlinear term xy the system (11) yields

$$\begin{aligned} x(t_{n+1}) &= c_0^{-\alpha} \left(-\sum_{j=1}^{n+1} c_j^\alpha x(t-jh) + a(y(t_n) - x(t_n)) \right) \\ y(t_{n+1}) &= \frac{-\sum_{j=1}^{n+1} c_j^\alpha y(t-jh) + (b - 2x(t_{n+1}))y(t_n)}{c_0^\alpha - x(t_{n+1})} \\ z(t_{n+1}) &= c_0^{-\alpha} \left(-\sum_{j=1}^{n+1} c_j^\alpha z(t-jh) + 2x(t_{n+1})y(t_n) - x(t_{n+1})y(t_{n+1}) - cz(t_n) \right) \end{aligned} \quad (12)$$

where $c_0^\alpha = h^{-\alpha}$, $x(t_0) = x_0, y(t_0) = y_0, z(t_0) = z_0$, and we choose $\varphi(h) = \sin(h)$ as a suitable function [34]. Conventionally when $\alpha = 1$, the system has two equilibrium points at $(0, 0, 0)$ and $(b, b, b^2/c)$ which depend on the parameters b and c only. The system exhibits chaotic behavior when the parameters set $(a, b, c) = (36.0, 28.0, 3.0)$. In the following simulations we will study the effect of the parameter α which does not affect the equilibrium points on the fractional order parameter α in order that chaotic responses appear. All the following simulations are performed using NSFD method, and when $b = 28.0$ and $c = 3.0$.

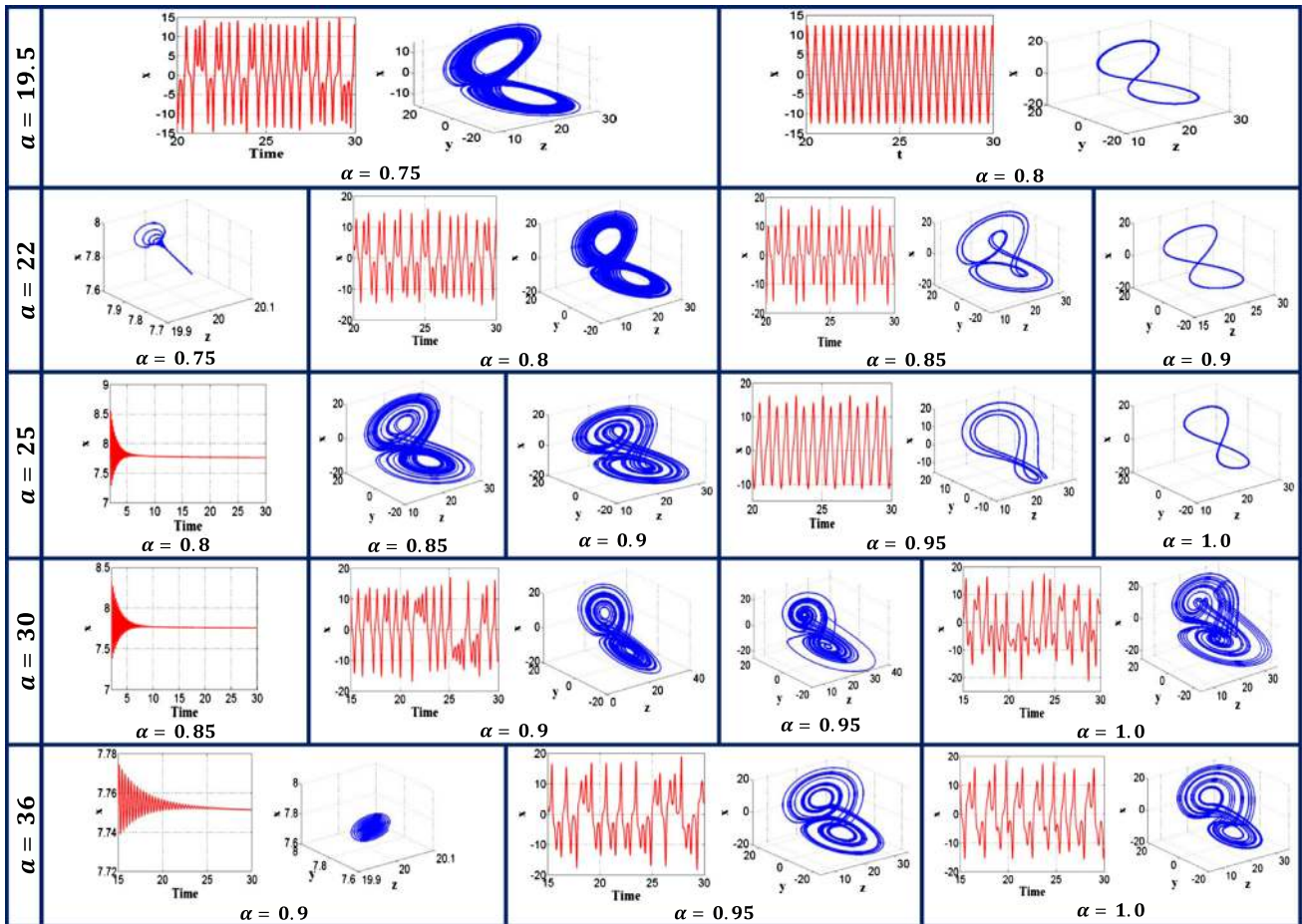


Fig. 1 The continuous responses of the Lü system versus the fractional-order α and parameter a .

Table 1 The Lü system performance versus the parameters (α , a).

	$a = 19.5$	$a = 22$	$a = 25$	$a = 30$	$a = 36$
$\alpha < 0.75$	Stable	Stable	Stable	Stable	Stable
$\alpha = 0.75$	Chaotic	Stable	Stable	Stable	Stable
$\alpha = 0.8$	Period 1	Chaotic	Stable	Stable	Stable
$\alpha = 0.85$	Period 1	Period 5	Chaotic	Stable	Stable
$\alpha = 0.9$	Period 1	Period 1	Chaotic	Chaotic	Stable
$\alpha = 0.95$	Period 1	Period 1	Period 3	Chaotic	Chaotic
$\alpha = 1.0$	Period 1	Period 1	Period 1	Chaotic	Chaotic

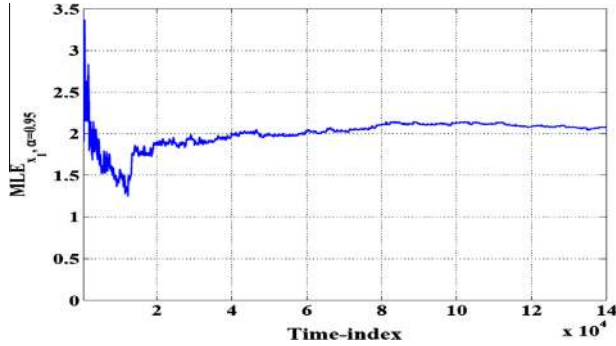
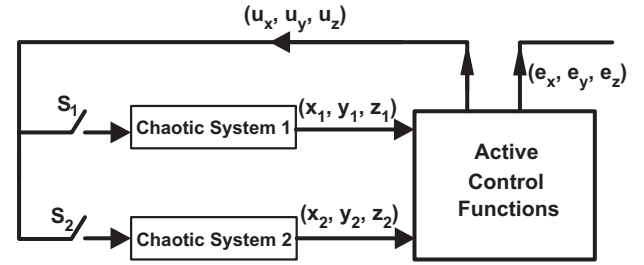
**Fig. 2** The time series calculation of the maximum Lyapunov exponent for the first system when $\alpha = 0.95$.

Fig. 1 shows the system responses when $a = 19.5$ for two different fractional orders. When α is less than 0.75 the system displays stable response. However, as α increases to 0.75, the system behaves chaotically. But for $\alpha = 0.8$ and higher orders, the system response is periodic with period 1. Therefore, the range of the fractional order α for chaotic behavior is $\alpha \in [0.75, 0.8)$. As the system parameter a increases to 22 and when $\alpha < 1$, the system responses pass by stable, chaotic, period-5, and period-1 responses when the fractional order α equals to 0.75, 0.8, 0.85, and 0.9 respectively as shown in **Fig. 1**. Therefore, the range of the fractional order α for chaotic response increases as the parameter a increases. Moreover, when the system parameter a increases to 25 and under the same values of the fractional order α , the range of chaotic response increases and different periodic attractors are obtained when the fractional order α belongs to the interval $[0.85, 0.95]$.

Similarly, when $a = 30$, the system becomes stable when α less than or equal to 0.85 and the chaotic response starts to appear in the range $[0.9, 1.0]$ while when $a = 36$, the system will be stable up to $\alpha = 0.9$ and the chaotic responses appear when $\alpha = 1.0$ which is the conventional case.

From **Fig. 1**, we can conclude the results in **Table 1**, where the chaotic responses appear for a wide range of the system parameter a , but in different ranges of the fractional order parameter α . Therefore, as the parameter a increases, the range of α for chaotic response increases and is shifted down. Moreover, it is expected that the Lü system can behave chaotically for larger values of $a > 36$ but with fractional order $\alpha > 1$. In addition, as the range of α increase, more cases of high-periodic responses will appear. As verification, the maximum Lyapunov exponent is calculated as approximately 2.08 as shown in **Fig. 2**. This calculation is based on using the nonlinear time series analysis of 150,000 points of x variable [36].

**Fig. 3** Block diagram of the proposed system.

Chaos synchronization between fractional order Lü and Newton–Leipnik systems

In this paper we provide a general technique for changing the response of any chaotic system to follow another chaotic pattern and this can be controlled through two switches as shown in **Fig. 3** which shows the general block diagram that describes the proposed technique. Assume two different chaotic systems, one of them is the master system, and the other is the slave. The purpose is to change the response of the slave system to synchronize with the master chaotic system via active control functions. These functions affect only the slave system without any loading on the master chaotic response.

The previous fractional order numerical technique will be applied on the Lü chaotic system defined by (11) with $a = 35$, $b = 28$, and $c = 3$, and the fractional-order Newton–Leipnik system defined by (13) as the other chaotic system with $(a_1, b_1, c_1) = (0.4, 0.4, 0.175)$.

$$\begin{aligned} D^\alpha x(t) &= -a_1 x(t) + y(t) + 10y(t)z(t) \\ D^\alpha y(t) &= -x(t) - b_1 y(t) + 5x(t)y(t) \\ D^\alpha z(t) &= c_1 z(t) - 5x(t)y(t) \end{aligned} \quad (13)$$

The minimum effective dimension for this system is 2.82 [37]. Assuming that the Lü system drives the Newton–Leipnik system, we define the drive (master) and response (slave) systems as follows

$$\begin{aligned} D^\alpha x_1(t) &= a(y_1(t) - x_1(t)) - S_1 u_x(t) \\ D^\alpha y_1(t) &= b y_1(t) - x_1(t)y_1(t) - S_1 u_y(t) \\ D^\alpha z_1(t) &= x_1(t)y_1(t) - c z_1(t) - S_1 u_z(t) \end{aligned} \quad (14)$$

and

$$\begin{aligned} D^\alpha x_2(t) &= -a_1 x_2(t) + y_2(t) + 10y_2(t)z_2(t) + S_2 u_x(t) \\ D^\alpha y_2(t) &= -x_2(t) - b_1 y_2(t) + 5x_2(t)y_2(t) + S_2 u_y(t) \\ D^\alpha z_2(t) &= c_1 z_2(t) - 5x_2(t)y_2(t) + S_2 u_z(t) \end{aligned} \quad (15)$$

where S_1 and S_2 are on–off parameters (digital bit) which either have the values “1” or “0” according to the required dependence between both systems as shown in **Fig. 3**. The unknown terms (u_x, u_y, u_z) in (14) and (15) are active control functions to be determined, and the error functions can be defined as:-

$$e_x = x_2(t) - x_1(t), e_y = y_2(t) - y_1(t), e_z = z_2(t) - z_1(t), \quad (16)$$

Eq. (16) together with (14) and (15) yield the error system

$$\begin{aligned}
D^\alpha e_x(t) &= -a_1(e_x(t) + x_1(t)) + (1 + 10(e_z(t) + z_1(t))) \\
&\quad \times (e_y(t) + y_1(t)) - a(y_1(t) - x_1(t)) + (S_1 + S_2)u_x(t) \\
D^\alpha e_y(t) &= -b_1e_y(t) - e_x(t) - x_1(t) - b_1y_1(t) + 5(e_z(t) + z_1(t)) \\
&\quad \times (e_x(t) + x_1(t)) - by_1(t) + x_1(t)z_1(t) + (S_1 + S_2)u_y(t) \\
D^\alpha e_z(t) &= c_1(e_z(t) + z_1(t)) + 5(e_y(t) + y_1(t))(e_x(t) + x_1(t)) \\
&\quad - x_1(t)y_1(t) + cz_1(t) + (S_1 + S_2)u_z(t) \quad (17)
\end{aligned}$$

We define active control functions $u_i(t)$ as

$$\begin{aligned}
(S_1 + S_2)u_x(t) &= V_x(e_x) - (1 + 10(e_z(t) + z_1(t)))(e_y(t) + y_1(t)) \\
&\quad + a(y_1(t) - x_1(t)) + a_1x_1(t) \\
(S_1 + S_2)u_y(t) &= V_y(e_y) + e_x(t) + x_1(t) + (b_1 + b)y_1(t) \\
&\quad - 5(e_z(t) + z_1(t))(e_x(t) + x_1(t)) - x_1(t)z_1(t) \\
(S_1 + S_2)u_z(t) &= V_z(e_z) - (c_1 + c)z_1(t) + 5(e_y(t) + y_1(t)) \\
&\quad \times (e_x(t) + x_1(t)) + x_1(t)y_1(t) \quad (18)
\end{aligned}$$

The terms V_x , V_y , and V_z are linear functions of the error terms e_x , e_y , e_z . With the choice of u_x , u_y , and u_z given by (18) the error system between the two chaotic systems (17) becomes

$$\begin{aligned}
D^\alpha e_x(t) &= -a_1e_x(t) + V_x(e_x(t)) \\
D^\alpha e_y(t) &= -b_1e_y(t) + V_y(e_y(t)) \\
D^\alpha e_z(t) &= c_1e_z(t) + V_z(e_z(t)) \quad (19)
\end{aligned}$$

In fact we do not need to solve (19) if the solution converges to zero. Therefore, the control terms $V_x(e_x)$, $V_y(e_y)$, and $V_z(e_z)$ can be chosen such that the system (20) becomes stable with zero steady state.

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = A \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}, \quad (20)$$

where A is a 3×3 real matrix, chosen so that all eigenvalues λ_i of the system (20) satisfy the following condition:

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2} \quad (21)$$

Then, by choosing the matrix A as follows:

$$A = \begin{pmatrix} a_1 - k & 0 & 0 \\ 0 & b_1 - k & 0 \\ 0 & 0 & -c_1 - k \end{pmatrix} \quad (22)$$

Then the eigenvalues of the linear system (18) are equal $(-k, -k, -k)$, which is enough to satisfy the necessary and sufficient condition (22) for all fractional orders $\alpha < 2$ [38]. In the following examples, we take $k = 1$ for simplicity.

Simulation results

The functions $\varphi_i(h)$, $i = 1, 2, 3$ are chosen according to the non-diagonal elements of the Jacobian matrix of the original continuous system of the error system

$$J_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (23)$$

Since $J_{ii} = -1$, then we choose $\varphi_i(h) = 1 - e^h$ for both system1 and system2 as a suitable function [34]. All the calculations of the two systems were numerically integrated using

the NSFD scheme with step size $h = 0.005$. Four different cases are discussed as follows:

- $(S_1, S_2) = (0, 0)$, then the two systems are working independently (no synchronization).
- $(S_1, S_2) = (0, 1)$, therefore the first system works normally without any loading effect, and the second system adapts its response to synchronize with the first system.
- $(S_1, S_2) = (1, 0)$, similarly the second system works individually, and the first system follows the second system exactly.
- Mixed mode synchronization case, where the switching parameters are a function of time.

Case 1: No synchronization $(S_1, S_2) = (0, 0)$

In this case, we validate the nonstandard finite difference method for the solution of both systems at $\alpha = 0.95$ and calculate the maximum Lyapunov exponent for the output. Fig. 4a shows the time domain response for the fractional order Lü system using the NSFD technique. The system has the faster response which is clear from the x , y , and z waveforms. The projection attractors in the xy , and xz planes with the 3D attractor are also introduced in Fig. 4a. Similarly, the time domain response and strange attractors of the second system (Newton–Leipnik) are shown in Fig. 4b. The time responses are very slow, and the attractors differ from the Lü system.

Case 2: system2 \rightarrow system1 synchronization when $(S_1, S_2) = (0, 1)$

In this case the Lü system works normally and the Newton–Leipnik system adapts its response to follow the Lü system. Fig. 5a shows the two system responses when $\alpha = 0.95$, the error function, and the active control signals versus time. The values of the x and z waveforms for system1 are represented by the solid lines however the dotted lines are the values of the x and z responses of system2. The error functions decay with time very fast as shown in Fig. 5a. These responses show the synchronization between the two systems when the initial conditions equal $(0.2, 0, 0.5)$ and $(0.9, 0, -0.3)$ for the systems (11) and (13) respectively. Although, the initial conditions are different system2 tracks system1 exactly. When $\alpha = 0.9$, system1 becomes stable $(x_1, y_1, z_1) = (x_2, y_2, z_2) = (-7.75, -7.75, 20)$. System2 synchronizes its response by the same way as shown in Fig. 5b. In this case, the control functions $(u_1, u_2, u_3) = (1554, 763.83, 296.6)$ when the initial conditions are $(-0.5, 0, 0.5)$ and $(1, 2, -0.5)$ respectively.

Case 3: system1 \rightarrow system2 synchronization when $(S_1, S_2) = (1, 0)$

When the switching parameters (S_1, S_2) are interchanged, no relation exists between the control variables and system2. In this case, the Lü system follows the behavior of the Newton–Leipnik system when the fractional order $\alpha = 0.95$. Fig. 5c and d illustrate the time domain responses and attractor projections in different planes for both systems. Although the initial points are different and apart, system1 adapts quickly to synchronize with system2 as shown in Fig. 5d.

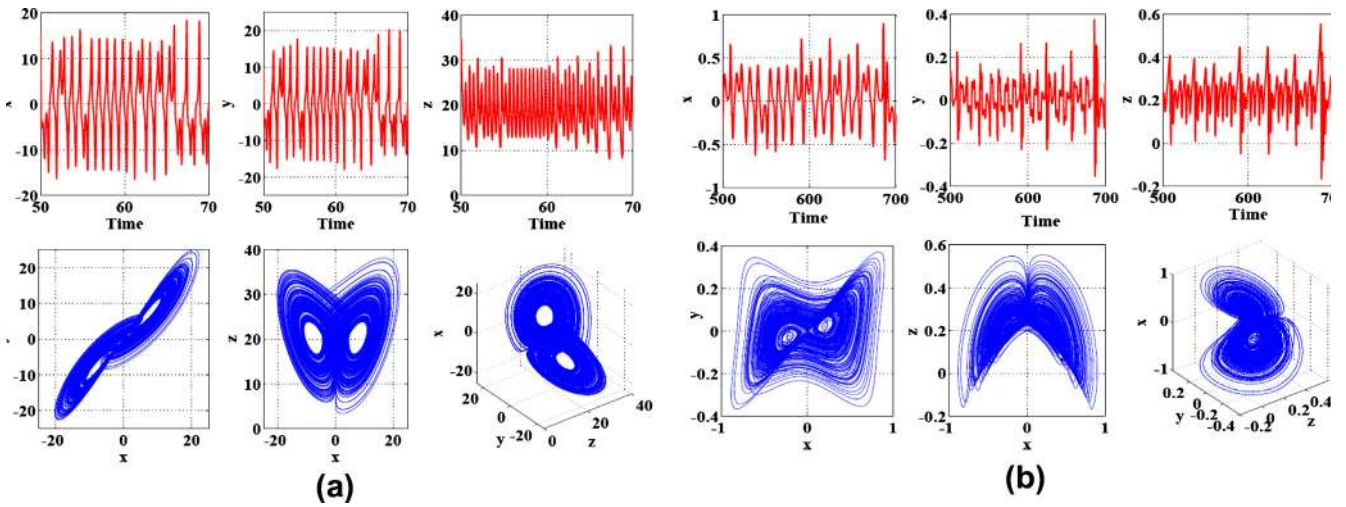


Fig. 4 Time domain waveforms and the strange attractors with $h = 0.005$ for (a) the first system under the initial condition $(0.2, 0.05)$ and (b) the second system when $\alpha = 0.95$ under the initial condition $(0.9, 0, -0.3)$.

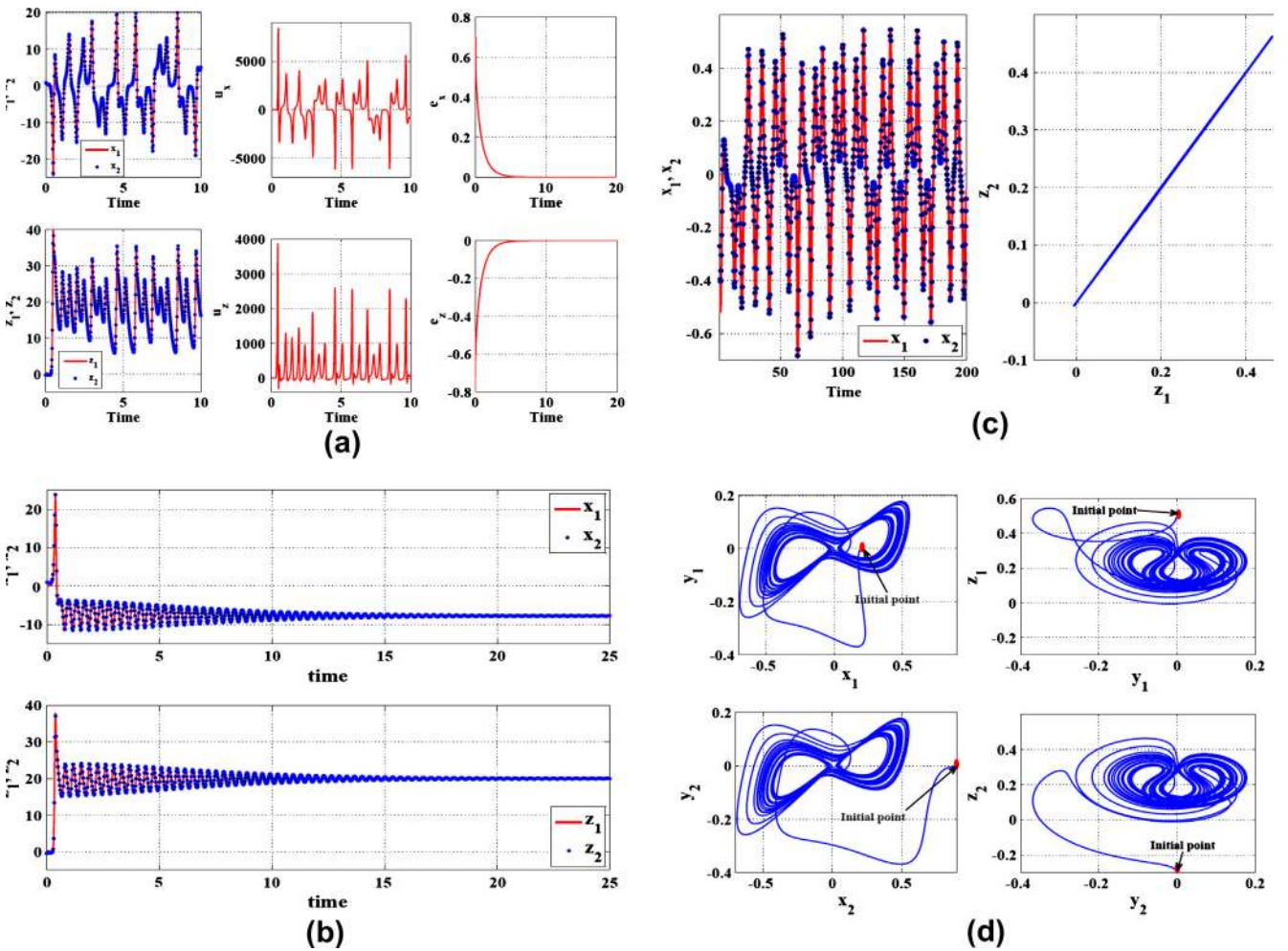


Fig. 5 (a) Time domain response for x_1, x_2, z_1 and z_2 the error functions and for both systems in case 2 with $h = 0.005$. (b) Time waveforms of x_1, x_2, z_1 and z_2 when $\alpha = 0.9$ where system2 follows system1 in the steady state for case 2, (c) the x_1, x_2 time waveforms and z_2 versus z_1 for case 3, and (d) the projection attractors of system1 and system2 when $\alpha = 0.95$ for case 3.

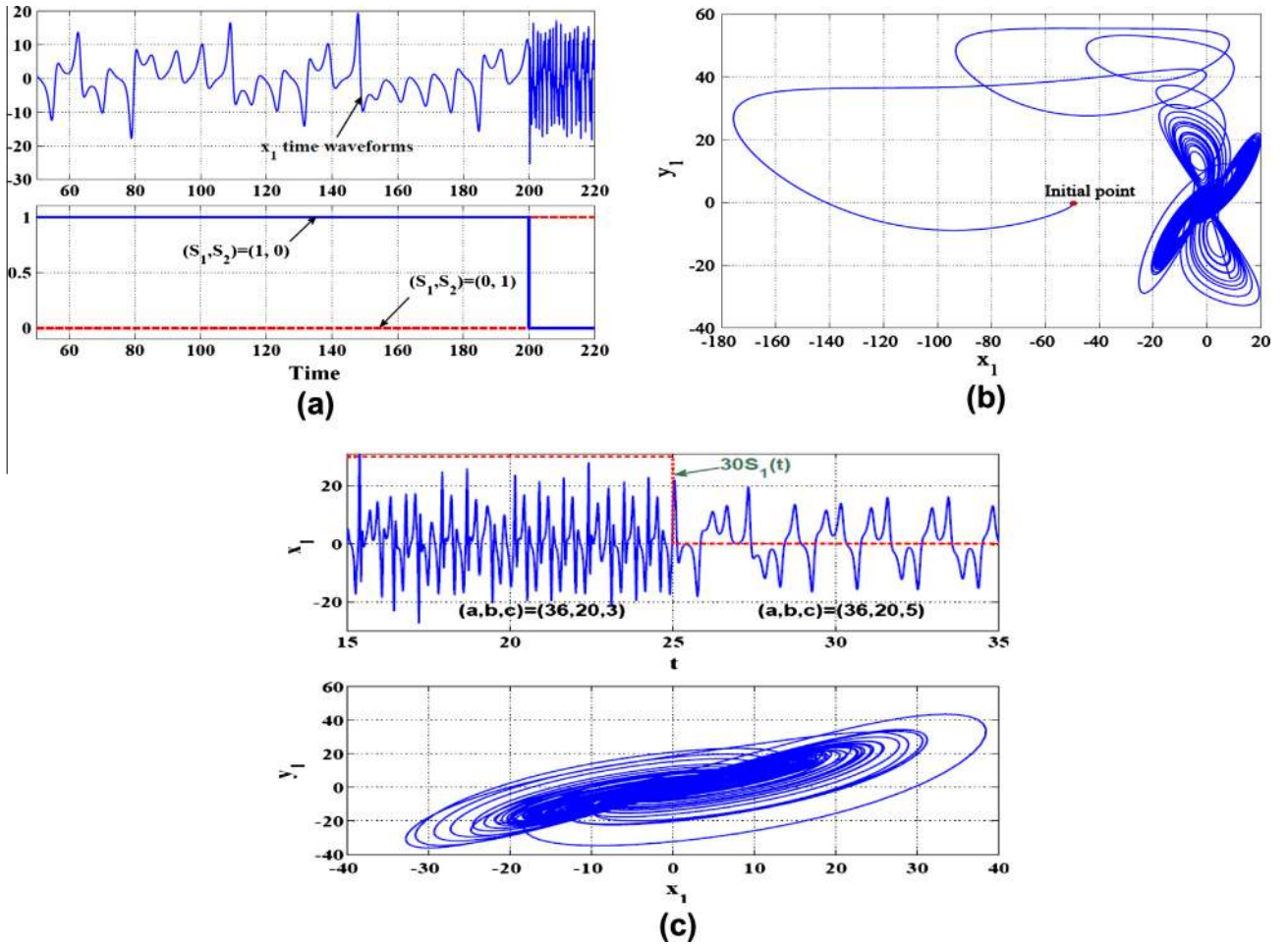


Fig. 6 (a) Time waveforms of x_1 , s_1 and s_2 of the mixed mode synchronization. (b) The xy projection attractor of the mixed mode synchronization for case 4 for two different systems, and (c) time waveforms and the x - y projection for two Lü systems with different parameters when $\alpha = 0.95$.

Case 4: mixed synchronizations

In this section, the values of (S_1, S_2) change with time, so we have mixed synchronizations.

$$(S_1, S_2) = \begin{cases} (1, 0) & t < 200 \text{ s} \\ (0, 1) & 200 \text{ s} < t < 400 \text{ s} \end{cases} \quad (24)$$

Therefore system2 will follow system1 in the first 200 s and then system1 will follow system2 in the last 200 s. But, due to the huge difference of amplitudes, we will multiply the output of system1 by 100 to make it in the same order for visualization. Fig. 6a shows the x_1 time waveforms in the interval $[0.85, 0.95]$. During the first 200 s x_1 is independent of system2 and hence the system output is very slow. However as the values of (S_1, S_2) interchange after $t = 200$ s the output x_1 synchronizes with system2 and then $x_1 = x_2$ at that interval shown in Fig. 6a. The transient response between the two cases is very fast, and the system behavior changes from slow response to accelerated response. The x - y projection of the response is shown in Fig. 6b, where the attractor changes from system1 into system2 smoothly.

The dynamic switching can be used also for the synchronization of two similar chaotic systems with different parameters. Fig. 6c shows the output x_1 versus time after modifying

the control functions (18) for two fractional order Lü systems with parameters $(a, b, c, \alpha) = (36, 20, 3, 0.95)$ and $(a, b, c, \alpha) = (36, 20, 5, 0.95)$ respectively. The switching parameters (S_1, S_2) equal to $(1, 0)$ in the first 25 s and $(0, 1)$ otherwise. It is clear that the speed of the system changes as the parameter c changes from 3 to 5 as shown from Fig. 6c and its x - y projection.

Conclusion

The first part of this paper discusses the smoothing change of the response from stable, periodic and chaotic as long as the parameters changes. The conclusion of this part shows us that the range of each response can be controlled by the system parameters or by the fractional-order parameters. Unlike the conventional synchronization techniques, the main objective of the second part is to discuss for the first time the switching synchronization between two different chaotic systems or one chaotic system with different parameters using the active control method. By using the proposed technique static synchronization (switching control independent of time), mono-dynamic synchronization (one of the control switches depends on time) or bi-dynamic synchronization (the two switches are time dependent). The concepts introduced in this paper have been

verified by using the fractional-order version of two different known chaotic systems which are the Lü and the Newton–Leipnik chaotic systems. Four different cases have been discussed together with the numerical techniques used to cover all the cases of the new block diagram introduced in this paper which is controlled by two switching parameters. These switching parameters can be a function of time to introduce a new concept of static and dynamic switching of synchronizations which makes the system more flexible as shown from the results. This technique can be used for the synchronization of many chaotic systems. All the numerical analysis have been done using the nonstandard finite difference method (NSFD) where the results indicated that the NSFD constructions are appropriate schemes because of the threshold and chaotic instabilities observed.

Conflict of interest

The authors have declared no conflict of interest.

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