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Computational algorithm for solving drug pharmacokinetic model under uncertainty with nonsingular kernel type Caputo-Fabrizio fractional derivative

Nesrine Harrouche [a], **Shaher Momani** [a,b], **Shatha Hasan** [c], **Mohammed Al-Smadi** [b,c,*]

[a] Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan
[b] Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, United Arab Emirates
[c] Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan

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Abstract This paper proposes an advanced numerical-analytical approach for handling a class of fuzzy fractional differential equations involving Caputo-Fabrizio derivative with a non-singular kernel arsing in the medical sector. The solution methodology relies on the reproducing-kernel algorithm to generate analytical solutions in the form of a uniformly convergent series in the direct sum of the desired Hilbert spaces. The effectiveness of the method is analyzed by studying some theoretical, analytical, and stability results of the derived solutions based on the reproducing kernel theory. Numerical simulations are also provided in tables and graphs to demonstrate the reliability of this algorithm in solving fuzzy models using the new Caputo-Fabrizio fractional operator, especially for the drug pharmacokinetic model. The obtained results show the ability of the applied algorithm to solve a wide range of nonlinear fractional models emerging in pharmacology, medicine, and biochemistry.

1. Introduction

Fuzzy differential equations appear in various fields of applied mathematics, engineering, physics, and many other areas. These equations have recently gained abundant attention due to their applications in artificial intelligence, image processing, pattern recognition, decision making, population models, particle systems, quantum optics, gravity, medicine, bioinformatics, and computational biology [1–5]. Historically, there are many approaches for fuzzy derivatives, and so for fuzzy differential equations. The common approaches are based on Hukuhara and Seikkala derivatives [4,5]. Lately, Bede defined the generalized differentiability of the fuzzy value function and then was applied to solve various types of equa-

* Corresponding author at: Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan.
E-mail address: mhm.smadi@bau.edu.jo (M. Al-Smadi).

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tions in which uncertainty could be addressed by integrating into dynamic systems [6]. The uncertainty during formulating such dynamical systems into mathematical models comes from many sources, such as data collection, initial data measurement, model parameters, model structure, errors in estimation values, and so on. It may appear in any part of the constructed differential equation, including the initial conditions (IC's), boundary conditions (BC's), parametric terms, or extra potential [7–10].

However, the topic of fractional calculus has gained the focus of many scholars due to its ability to describe several complex phenomena. It is not only a productive and emerging field, but also a new philosophy to apply non-local operators to real world problems. Expectedly, studying models with uncertainty behaviours by the use of fractional non-local operators may lead to more interesting and accurate results. As a result, Agarwal et al. [11] used the concept of Hukuhara derivative to introduce the fuzzy Riemann–Liouville fractional differential equations which was the starting point of the topic in fuzzy fractional calculus. Many research works demonstrated that fractional derivatives succeeds in describing systems in which memory plays a significant role. Unfortunately, the well known fractional derivatives, Riemann–Liouville and Caputo have some disadvantages such as the physically unacceptable IC's in Riemann–Liouville derivative and that both of them have singularity in their kernels and consequently cannot give accurate description to the full effect of the memory [12–18]. Due to these disadvantages, Caputo and Fabrizio suggested a fractional derivative operator with exponential non-singular kernel [19].

The interest of Caputo-Fabrizio approach is due to its capability to describe variants and structures of varying scales that cannot be well visualized through classical local derivatives and fractional operators having a singular kernel. Therefore, many researchers have adopted Caputo-Fabrizio approach in their studies and mathematical analysis. Whereas some properties and applications of this new fractional concept related to resistance, inductance, capacitance and propagation circuit are given in [20–26] and the references therein. Rezapour et al. [20] studied and analyzed anthrax disease of animals based in view of Caputo-Fabrizio fractional derivative. In [21], a novel operator, the infinite coefficient-symmetric Caputo-Fabrizio derivative, has been introduced to deal with approximate solutions of fractional integrodifferential equations. Using this concept, the homotopy analysis transform method [22] has been profitably implemented on a new fractional model for human liver. While Mohammadi et al. [23] proposed a new analysis of the mathematical modeling of hearing loss due to the Mumps virus involving the Caputo-Fabrizio factor. In [27], fractional differential equations with uncertainty based on this new operator was studied and applied to some real world problems as the falling body problem and decay problem. Anyhow, introducing a straightforward formula for such problems is a sophisticated process. Therefore, treating the complexities of uncertain data is necessary. Using the parametrization of fuzzy numbers, the fuzzy fractional problem can be converted to a crisp system of fractional equations, which are solved approximately. Several advanced numerical and semi-analytical methods have been used for obtaining analytic and approximate solutions to many fractional problems, for more details, we refer to [28–33]. This paper will add an efficient and reliable numerical technique, namely the reproducing kernel algorithm, to approximate the solution of fuzzy fractional initial value problems (IVPs) involving Caputo-Fabrizio fractional derivative under the generalized Hukuhara differentiability concept. This method is considered as one of the most powerful methods for solving both linear and non-linear equations, which has important applications in numerical analysis, applied mathematics, and theoretical physics, see [34–36]. Nevertheless, in the present work, we consider the fuzzy fractional IVP of the form:

$$ {}^{CF}D^{\alpha}_{a^+}y(x) = G(x, y(x)), \quad x \in [a, b], \quad 0 < \alpha \leqslant 1, \tag{1.1} $$

along with fuzzy initial condition

$$ y(a) = \beta, \tag{1.2} $$

where $\beta$is an arbitrary fuzzy number, α is the order of the fractional derivative, ${}^{CF}D^{\alpha}_{a^+}$ denotes the Caputo-Fabrizio fractional derivative of order $\alpha, G(x, y(x))$ is linear or nonlinear in term of y, and $y(x)$ is unknown fuzzy-valued function to be determined.

In general, there is no classic method that produces a accurate solution in a closed form for fuzzy-fractional types of medical systems. Therefore, efficient and advanced methods of exploring solutions of such models are urgently needed, which gives us the impetus to search for numerical solutions. Motivated by the above discussion, this analysis aims to design a novel iterative method to generate the analytic and approximate solutions of fuzzy fractional drug pharmacokinetic models by employing a new fractional operator in terms of non-singular kernel, Caputo-Fabrizio. In this work, several reproducing functions are introduced in Hilbert spaces to generate a complete normal system. Meanwhile, approximate solutions uniformly converge to analytical solutions in the direct sum of these spaces. Stability and error analysis are discussed as well. Finally, two numerical examples are tested to verify the accuracy, efficiency, and reliability of the novel algorithm. The rest of the paper is divided as follows. In Section 2, we give preliminaries of fractional calculus, fuzzy set, and fuzzy fractional derivative. In Section 3, we present the basic definitions and theorems concerning reproducing kernel Hilbert space. The reproducing kernel algorithm is applied for obtaining the analytical solutions of the fuzzy fractional IVP (1.1) and (1.2) in Section 4. In Section 5, stability and convergence analysis are also studied. To detect the validity and effectiveness of the suggested algorithm, we present some numeric experiments in Section 6, including an application to the fractional pharmacokinetics model. Meanwhile, We present the results in tables and figures to see the effect of the Caputo-Fabrizio derivative to the considered model. Finally, the conclusion will be drawn in the last section.

2. Basics interrelated to fractional and fuzzy calculus

In this section, we introduce some necessary definitions which will be used throughout the paper. The study of differentiation and integration of uncertain functions is called fuzzy calculus. This branch of mathematics is an efficient tool for robust modelling of many real world phenomena. A basic concept in the fuzzy theory is the fuzzy number which can be considered as a generalization of a real (crisp) number in the sense that it doesn't refer to one single value but to a set of values where each possible value has its own weight between 0 and 1. This weight is called the membership function. A more formal definition for a fuzzy number is given below.

Definition 2.1. [37] A fuzzy number $\upsilon : \mathbb{R} \longrightarrow [0,1]$ is a fuzzy subset of $\mathbb{R}$which satisfies:

- υ is normal, i.e., there is at least one element $\tau_0 \in \mathbb{R}$ whose degree of membership equals to 1.
- υ is fuzzy convex, i.e., $\upsilon(\lambda\eta + (1-\lambda)\zeta) \geqslant \min(\upsilon(\eta), \upsilon(\zeta))$ for each $\eta, \zeta \in \mathbb{R}$ and $0 \leqslant \lambda \leq 1$.
- υ is uppersemicontinuous, i.e., the set$[\upsilon]^r = \begin{cases} \overline{\{x \in \mathbb{R} : \upsilon(x) \geqslant r\}}, & 0 < r \leqslant 1, \\ \overline{\{x \in \mathbb{R} : \upsilon(x) \geqslant 0\}}, & r = 0 \end{cases}$ is closed for each $r \in [0,1]$.
- The support of υ, i.e., $[\upsilon]^0$, is bounded.

The symbol $[\upsilon]^r$ is called the r-level representation for a fuzzy number υ and $[\upsilon]^1$ is called the core of υ. So, using Definition 2.1, υ will be a fuzzy number whenever $[\upsilon]^r$ be a convex and compact set of $\mathbb{R}$ for all r in $[0,1]$ and $[\upsilon]^1 \neq \emptyset$. Moreover, let υ be a fuzzy number, then $[\upsilon]^r = [\upsilon_1(r), \upsilon_2(r)]$, in which $\upsilon_1(r) = \upsilon_{1r} = \min\{x; x \in [\upsilon]^r\}$ and $\upsilon_2(r) = \upsilon_{2r} = \max\{x; x \in [\upsilon]^r\}$ for all r in $[0,1]$. The following theorem is a basic rule in fuzzy numbers theory:

Theorem 2.2. [38] *Assume that* $\upsilon_1, \upsilon_2 : [0,1] \longrightarrow \mathbb{R}$ *fulfill the following terms:*

- *The functions* υ_1 *and* υ_2 *are bounded non-decreasing and non-increasing, respectively. Provided that* $\upsilon_1(1) \leqslant \upsilon_2(1)$.
- *For all* $k \in (0,1], \lim_{r\longrightarrow k^-} \upsilon_1(r) = \upsilon_1(k)$ *and* $\lim_{r\longrightarrow 0^+} \upsilon_1(r) = \upsilon_1(0)$.
- *For all* $k \in (0,1], \lim_{r\longrightarrow k^-} \upsilon_2(r) = \upsilon_2(k)$ *and* $\lim_{r\longrightarrow 0^+} \upsilon_2(r) = \upsilon_2(0)$.

Then, $\upsilon : \mathbb{R} \longrightarrow [0,1]$ *which is given as* $\upsilon(x) = \sup\{r; \upsilon_1(r) \leqslant x \leqslant \upsilon_2(r)\}$ *be a fuzzy number in terms of parameterization* $[\upsilon_1(r), \upsilon_2(r)]$. *Besides, let* υ *be a fuzzy number in terms of parameterization* $[\upsilon_1(r), \upsilon_2(r)]$, *then the functions* υ_1 *and* υ_2 *satisfy the above-mentioned conditions.*

In general, the arbitrary fuzzy number υ can be presented as an ordered pair of functions (υ_1, υ_2) which satisfies Theorem 2.2. For convenience, we will write υ_{1r} and υ_{2r} as alternatives to $\upsilon_1(r)$ and $\upsilon_2(r)$, respectively. This representation can be used to present the most common mathematical operations on arbitrary fuzzy numbers $w = (w_{1r}, w_{2r}), v = (v_{1r}, v_{2r})$ and $\gamma \in \mathbb{R} \setminus \{0\}$ as follows:

- $[w+v]^r = [w]^r + [v]^r = [w_{1r} + v_{1r}, w_{2r} + v_{2r}]$,
- $[\gamma w]^r = \gamma[w]^r = [\min\{\gamma w_{1r}, \gamma w_{2r}\}, \max\{\gamma w_{1r}, \gamma w_{2r}\}]$.

Moreover, $v = w$ iff $[v]^r = [w]^r$, that is $v_{1r} = w_{1r}$ and $v_{2r} = w_{2r}$. In the rest of this paper, we use the symbol $\mathbb{R}_F$ to indicate the set of fuzzy numbers on $\mathbb{R}$.

Definition 2.3. [4] Let $v, w \in \mathbb{R}_F$. If there exists an element $\theta \in \mathbb{R}_F$ such that $w = v + \theta$, then we say that θ is the H-difference of w and v, which denoted by $w \ominus v$.

Remark 2.4. Although the sign $\ominus$ represents Hukuhara difference, it should be noted that $w \ominus v \neq w + (-1)v$ but $w + (-1)v = w - v$. Moreover, $[w \ominus v]^r = [w_1(r) - v_1(r), w_2(r) - v_2(r)]$, if the H-difference $w \ominus v$ exists. It has been found that the length of support has been increased by Hukuhara differentiable function. In [6], Bede and Stefanini introduced the generalized Hukuhara difference (gH-difference) as

$$u \ominus_{gH} w = v \Longleftrightarrow u = v + w \ or \ v = u + (-1)w,$$

with r-cuts

$$[u \ominus_{gH} v]^r = [min\{u_{1r} - v_{1r}, u_{2r} - v_{2r}\}, max\{u_{1r} - v_{1r}, u_{2r} - v_{2r}\}].$$

Definition 2.5. [6] The complete metric structure over $\mathbb{R}_F$ can be given using the mapping $D_H : \mathbb{R}_F \times \mathbb{R}_F \longrightarrow \mathbb{R}^+ \bigcup\{0\}$ such that

$$D_H(v, w) = \sup_{0 \leqslant r \leqslant 1} \max\{| v_{1r} - w_{1r} |, | v_{2r} - w_{2r} |\},$$

for any fuzzy numbers $v = (v_{1r}, v_{2r})$ and $w = (w_{1r}, w_{2r})$, where D_H is the Hausdorff distance.

A fuzzy number Ω is said to be a triangular fuzzy number (TFN) and denoted by (a_1, a_2, a_3), where $a_1, a_2, a_3 \in \mathbb{R}$ and $a_1 < a_2 < a_3$, if its membership function has the form [39]:

$$\Omega(z) = \begin{cases} \frac{z - a_1}{a_2 - a_1}, & if \ a_1 < z \leqslant a_2, \\ \frac{a_3 - z}{a_3 - a_2}, & if \ a_2 < z < a_3, \\ 0, & otherwise. \end{cases}$$

The r-cut of the TFN $\Omega = (a_1, a_2, a_3)$ is the closed interval so that

$$[\Omega]^r = [\Omega_{1r}, \Omega_{2r}] = [(a_2 - a_1)r + a_1, a_3 - (a_3 - a_2)r], \ \ r \in (0,1].$$

A basic concept in fuzzy calculus is the concept "fuzzy function". Anyhow, the fuzzy function y over $[a,b]$ is a mapping $y : [a,b] \longrightarrow \mathbb{R}_F$. It is continuous at $\tau_0 \in [a,b]$ if for every $\varepsilon > 0, \exists\delta = \delta(\tau_0, \varepsilon) > 0$ such that $D_H(y(\tau), y(\tau_0)) < \varepsilon$, for each $\tau \in [a,b]$, whenever $| \tau - \tau_0 |< \delta$[39].

Definition 2.6. [6] Let $y : [a,b] \longrightarrow \mathbb{R}_F$and $\tau_0 \in [a,b]$. Then, we say that y is strongly generalized differentiable at τ_0, if there exists an element $y'(\tau_0) \in \mathbb{R}_F$ such that

- For each $h > 0$, the H-differences $y(\tau_0 + h) \ominus y(\tau_0)$ and $y(\tau_0) \ominus y(\tau_0 - h)$ exist and defined as

$$\lim_{h\longrightarrow 0^+} \frac{y(\tau_0 + h) \ominus y(\tau_0)}{h} = \lim_{h\longrightarrow 0^+} \frac{y(\tau_0) \ominus y(\tau_0 - h)}{h} = y'(\tau_0),$$

- For each $h > 0$, the H-differences $y(\tau_0) \ominus y(\tau_0 + h)$ and $y(\tau_0 - h) \ominus y(\tau_0)$ exist and defined as

$$\lim_{h\longrightarrow 0^+} \frac{y(\tau_0 + h) \ominus y(\tau_0)}{-h} = \lim_{h\longrightarrow 0^+} \frac{y(\tau_0) \ominus y(\tau_0 - h)}{-h} = y'(\tau_0).$$

Here, the limits are evaluated in the metric space $(\mathbb{R}_F, D)$.

Nevertheless, the function y is called i-differentiable for $i = 1, 2$, when y is the strongly generalized differentiable in the i^{th}-direction, in which the i^{th}-derivative of y at τ_0 is given by $y'(\tau_0) = D_i y(\tau_0)$. In the following, we present the main concepts and theories that enable us to solve fuzzy fractional IVPs in Caputo-Fabrizio sense. To do so, we will use the following notations: $H^1(a,b) = \Big\{g : [a,b] \longrightarrow \mathbb{R} : \int_a^b (g(x))^2 dx < \infty,$ $\int_a^b (g'(x))^2 dx < \infty\}, C([a,b], \mathbb{R}_F)$ refers to all continuous fuzzy valued functions on the interval $[a,b]$, and $L^1([a,b], \mathbb{R}_F) = \{F : [a,b] \leftrightarrow \mathbb{R}_F$ is measurable function such that $\int_a^b D(F(x), 0)dx < \infty\Big\}$.

Definition 2.7. [40] Given $g \in H^1(a,b)$ and $0 < \alpha \leqslant 1$, then the Caputo fractional derivative of $g$of order $\alpha$is defined as

$$ {}^{C}D^{\alpha}_{a+}g(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-\zeta)^{-\alpha} g'(\zeta) d\zeta, \quad x > a. \tag{2.1}$$

Replacing the kernel $(x-\zeta)^{-\alpha}$by the function $\exp\left(\frac{-\alpha}{1-\alpha}(x-\zeta)\right)$ and $\frac{1}{\Gamma(1-\alpha)}$by $\frac{M(\alpha)}{1-\alpha}$, we obtain the new definition of Caputo-Fabrizio fractional derivative of order αsuch that

$$ {}^{CF}D^{\alpha}_{a+}g(x) = \frac{M(\alpha)}{1-\alpha} \int_a^x e^{-\frac{\alpha}{1-\alpha}(x-\zeta)} g'(\zeta) d\zeta, \quad x \geqslant a, \tag{2.2}$$

in which $M(\alpha)$ is a normalization function so that $M(0) = M(1) = 1$.

The definition of Caputo-Fabrizio derivative for fuzzy valued functions is characterized as follows.

Definition 2.8. [41] Let $g : [a,b] \rightarrow \mathbb{R}_F$ be a generalized differentiable with $g' \in C([a,b], \mathbb{R}_F) \cap L^1((a,b), \mathbb{R}_F)$. Then, the generalized fuzzy Caputo-Fabrizio fractional derivative of order $\alpha, \alpha \in (0,1]$ is

$$ {}^{CF}D^{\alpha}_{a+}g(x) = \frac{M(\alpha)}{1-\alpha} \int_a^x e^{-\frac{\alpha}{1-\alpha}(x-\zeta)} g'(\zeta) d\zeta, x \geqslant a. \tag{2.3}$$

Definition 2.9. [41] The generalized differentiable fuzzy valued function $g : [a,b] \rightarrow \mathbb{R}_F$ with $g' \in C((a,b], \mathbb{R}_F) \cap L^1((a,b), \mathbb{R}_F)$ is said to be ${}^{CF}[(1)-\alpha]$-differentiable whenever $\left[{}^{CF}D^{\alpha}_{a+}g(x)\right]^r = \left[{}^{CF}D^{\alpha}_{a+}g_{1r}(x), {}^{CF}D^{\alpha}_{a+}g_{2r}(x)\right]$. While if $\left[{}^{CF}D^{\alpha}_{a+}g(x)\right]^r = \left[{}^{CF}D^{\alpha}_{a+}g_{2r}(x), {}^{CF}D^{\alpha}_{a+}g_{1r}(x)\right]$, then g is said to be ${}^{CF}[(2)-\alpha]$-differentiable.

Theorem 2.10. [41] *Let $g : [a,b] \rightarrow \mathbb{R}_F$ be generalized differentiable at t_0 with $g' \in C((a,b], \mathbb{R}_F) \cap L^1((a,b), \mathbb{R}_F)$. Then, g is ${}^{CF}[(1)-\alpha]$-differentiable iff g is (1)-differentiable while g is ${}^{CF}[(2)-\alpha]$-differentiable iff g is (2)-differentiable.*

Theorem 2.11. *Consider the fuzzy fractional IVP:*

$$ {}^{CF}D^{\alpha}_{a+}y(x) = G(x, y(x)), \quad 0 < \alpha \leqslant 1, \quad x \in [a,b], \tag{2.4}$$

$$y(a) = \beta \in \mathbb{R}_F,$$

where $G : [a,b] \times \mathbb{R}_F \longrightarrow \mathbb{R}_F$ satisfies the following conditions:

a) $[G(x,y(x))]^r = [G_{1r}(x, y_{1r}(x), y_{2r}(x)), G_{2r}(x, y_{1r}(x), y_{2r}(x))]$.
b) *For* $\varepsilon > 0, \exists \delta > 0$ *so that* $\forall r \in [0,1]$, *we have* $| G_{1r}(x,y,z) - G_{1r}(x_1,y_1,z_1) | < \varepsilon$ and $| G_{2r}(x,y,z) - G_{2r}(x_1,y_1,z_1) | < \varepsilon$, *where* $(x,y,z), (x_1,y_1,z_1) \in G[x_0, x_0+a] \times \mathbb{R}^2, \|(x,y,z) - (x_1,y_1,z_1)\|_{\mathbb{R}^3} < \delta$, *in which G_{1r} and G_{2r} are uniformly bounded.*
c) *There is a positive number L so that for all $r \in [0,1]$, we have* $| G_{1r}(x,y,z) - G_{1r}(x_1,y_1,z_1) | \leqslant L \max\{| y_2 - y_1 |, | z_2 - z_1 |\}$ *and* $| G_{2r}(x,y,z) - G_{2r}(x_1,y_1,z_1) | \leqslant L \max\{| y_2 - y_1 |, | z_2 - z_1 |\}$.

Let $y(x)$ be ${}^{CF}[(1)-\alpha]$-differentiable, then the fuzzy fractional IVP (2.4) can be converted into the following system of fractional ODEs:

$$ {}^{CF}\left(D^{\alpha}_{x_{a+}} y_{1r}\right)(x) = G_{1r}(x, y_{1r}(x), y_{2r}(x)), \tag{2.5}$$

$$ {}^{CF}\left(D^{\alpha}_{x_{a+}} y_{2r}\right)(x) = G_{2r}(x, y_{1r}(x), y_{2r}(x)),$$

$$y_{1r}(a) = \beta_{1r},$$

$$y_{2r}(a) = \beta_{2r},$$

and let $y(x)$ be ${}^{CF}[(2)-\alpha]$-differentiable, then the fuzzy fractional IVP (2.4) can be converted into the following system of fractional ODEs:

$$ {}^{CF}\left(D^{\alpha}_{x_{a+}} y_{1r}\right)(x) = G_{2r}(x, y_{1r}(x), y_{2r}(x)), \tag{2.6}$$

$$ {}^{CF}\left(D^{\alpha}_{x_{a+}} y_{2r}\right)(x) = G_{1r}(x, y_{1r}(x), y_{2r}(x)),$$

$$y_{1r}(a) = \beta_{1r},$$

$$y_{2r}(a) = \beta_{2r}.$$

Proof. From (b), we have G_{1r}, G_{2r} are equicontinuous which implies that G is continuous. Due to $D(G(x,y), G(x,z)) = Sup_{r\in[0,1]} max\{|(G_{1r}(x,y_{1r}, \; y_{2r}) - G_{1r}(x,z_{1r},z_{2r})|, |G_{2r}(x,y_{1r},z_{2r}) - G_{2r}(x,y_{1r},z_{2r})|\} \leqslant \kappa Sup_{r\in[0,1]} max\{| y_{1r} - z_{1r}) |, | y_{2r} - z_{2r}) |\} = \kappa D(y,z)$ with the aid of (c) that ensures that G fulfills Lipschitz condition, and the boundedness of G utilizing (b), it follows that the solution of fractional ODEs (2.5) is unique. Now, since the solution is ${}^{CF}[(1)-\alpha]$-differentiable, so both of G_{1r} and G_{2r} are ${}^{CF}[(1)-\alpha]$-differentiable. Consequently, $\{y_{1r}(x), y_{2r}(x)\}$is a solution of (2.5). On the other hand, let $\{y_{1r}(x), y_{2r}(x)\}$be a solution of (2.4), then the existence and uniqueness of fuzzy solution y is achieved with the aid of Lipschitz property. Anyhow, y is ${}^{CF}[(1)-\alpha]$-differentiable, so $[y(x)]^r = [y_{1r}(x), y_{2r}(x)]$ is a solution of (2.4). By uniqueness of the solution, we get that IVPs (2.4) and (2.5) are equivalent. □

3. Reproducing kernel hilbert space method

In this section, we recall the definitions of the Hilbert spaces $H_2^1[a,b]$ and $H_2^2[a,b]$, define reproducing kernel function and construct orthonormal basis on the space of the direct sum $H_2^2[a,b] \oplus H_2^2[a,b]$. After that, we introduce the RKHS method and construct an algorithm for obtaining the approximate solution of a general first order fuzzy differential equation that converges to the analytic solution.

Definition 3.1. [42] Let Ebe a non-empty set. A function $R : E \times E \longrightarrow \mathbb{C}$ is called a reproducing kernel (RK) of a Hilbert space H if

a) For all $x \in E$, we have $R(.,x) \in H$.
b) For all $x \in E$ and $\varphi \in H$, $\langle \varphi(.)$, we have $R(.,x)\rangle = \varphi(x)$.

The second property is said the reproducing property, which indicates that the value of φ at x is reproduced by the inner product of $\varphi(.)$and $K(.,x)$. While the Hilbert space that contains a reproducing-kernel is said to be the reproducing-kernel Hilbert space (RKHS) [43–47].

Definition 3.2. [43] The Hilbert space $H_2^2[a,b]$is defined as $H_2^2[a,b] = \{y : y, y'' \in AC[a,b], y'' \in L^2[a,b], y(a) = 0\}$ with inner product given by

$$\left\langle y_1(x), y_2(x)\right\rangle_{H_2^2} = y_1(a)y_2(a) + y_1'(a)y_1'(a) + \int_a^b y_1''(x)y_2''(x)dx, \tag{3.1}$$

and norm $\|y_1\|_{H_2^2} = \sqrt{\left\langle y_1(x), y_1(x)\right\rangle_{H_2^2}}$, for $y_1, y_2 \in H_2^2[a,b]$.

Theorem 3.3. [39] *The space $H_2^2[a,b]$ is a complete RKHS with reproducing kernel $N_x(s)$ such that*

$$N_x(s) = \begin{cases} \frac{1}{6}(s-a)(2a^2 - s^2 + 3x(2+s) - a(6+s+3x)), & s \leqslant x, \\ \frac{1}{6}(x-a)(2a^2 - x^2 + 3s(2+x) - a(6+x+3s)), & s > x. \end{cases} \tag{3.2}$$

Definition 3.4. [43] The inner product space $H_2^1[a,b]$ is defined as $H_2^1[a,b] = \{y(x) : y \in AC[a,b] \text{ and } y' \in L^2[a,b]\}$. Its inner product and the norm in $H_2^1[a,b]$ are given by

$$\left\langle y_1(x), y_2(x)\right\rangle_{H_2^1} = \int_a^b y_1'(x)y_2'(x) + y_1(x)y_2(x)dx, \tag{3.3}$$

and $\|y_1\|_{H_2^1} = \sqrt{\left\langle y_1(x), y_1(x)\right\rangle_{H_2^1}}$, respectively, where $y_1, y_2 \in H_2^1[a,b]$.

Theorem 3.5. [43] *The space $H_2^1[a,b]$ is complete kernel space whose RK is $M_x(s)$ such that*

$$M_x(s) = \frac{1}{2\sinh(b-a)} \times [\cosh(x+s-b-a) + \cosh(|x-s| - b + a)]. \tag{3.4}$$

Reproducing kernel functions possess some important properties such as being symmetric, unique, and non-negative. For solving fuzzy fractional IVPs, the following spaces are required.

Definition 3.6. [45] Let $H^i[a,b], i = 1,2$, be the Hilbert spaces that can be defined as direct sums of $H_2^1[a,b]$ and $H_2^2[a,b]$ such that $H^i[a,b] = H_2^i[a,b] \oplus H_2^i[a,b] = \left\{(y_1(t), y_2(t))^T : y_1, y_2 \in H_2^i[a,b]\right\}$ equipped with the inner product and norm of $y(t) = (y_1(t), y_2(t))^T, \omega = (\omega_1(t), \omega_2(t))^T$as follows $\left\langle y(t), \omega(t)\right\rangle_{H^i} = \left\langle y_1(t), \omega_1(t)\right\rangle_{H_2^1} + \left\langle y_2(t), \omega_2(t)\right\rangle_{H_2^2}$ and $\|y\|_{H^i} = \sqrt{\|y_1\|_{H_2^i}^2 + \|y_2\|_{H_2^i}^2}$, respectively.

4. Solutions of FFIVPs under generalized CF differentiability

In this section, analytic and approximate solutions of (1.1) and (1.2) are given in Hilbert space $H^2[a,b]$ by constructing an orthonormal basis of $H^i[a,b]$ relies on the Gram-Schmidt procedures with respect to ${}^{CF}[(1)-\alpha]$-differentiability or with respect to ${}^{CF}[(2)-\alpha]$-differentiability.

As a first step, we construct the linear operator L_{jr}as $L_{jr} : H_2^2[a,b] \longrightarrow H_2^1[a,b]$such that $L_{jr}y_{jr}(x) = \left({}^{CF}D_{a+}^{\alpha}y_{jr}\right)(x)$, $j = 1,2$. Put $G_r = (G_{1r}, G_{2r})^\top, y_r = (y_{1r}, y_{2r})^\top, \beta_r = (\beta_{1r}, \beta_{2r})^\top$, and $L_r = diag(L_{1r}, L_{2r})$, where $L_r : H^2[a,b] \longrightarrow H^1[a,b]$, and use the transform $z_{1r}(x) = y_{1r}(x) - \beta_{1r}$and $z_{2r}(x) = y_{2r}(x) - \beta_{2r}$ together with these notations, we can convert the system into the form

$$L_r z_r(x) = G_r(x, z_r(x)) = G_r(x, z_{1r}(x), z_{2r}(x)), \tag{4.1}$$

with the homogeneous initial condition

$$z_r(a) = 0. \tag{4.2}$$

Lemma 4.1. *$L_r : H^2[a,b] \longrightarrow H^1[a,b]$ is linear and bounded operator.*

Proof. For linearity, let α_1 and α_2 be any constants and $z_1 = [z_{11}, z_{12}]^T, z_2 = [z_{21}, z_{22}]^T \in H^2[a,b]$. Then, we have

$$\begin{aligned} L_r(\alpha_1 z_1 + \alpha_2 z_2) &= L_r\left([\alpha_1 z_{11} + \alpha_2 z_{21} \quad \alpha_1 z_{12} + \alpha_2 z_{22}]^T\right) \\ &= \begin{bmatrix} L_{1r} & 0 \\ 0 & L_{2r} \end{bmatrix} [\alpha_1 z_{11} + \alpha_2 z_{21} \quad \alpha_1 z_{12} + \alpha_2 z_{22}] \\ &= \begin{bmatrix} L_{1r}(\alpha_1 z_{11} + \alpha_2 z_{21}) \\ L_{2r}(\alpha_1 z_{12} + \alpha_2 z_{22}) \end{bmatrix} = \begin{bmatrix} {}^{CF}D_a^{\alpha}(\alpha_1 z_{11} + \alpha_2 z_{21}) \\ {}^{CF}D_a^{\alpha}(\alpha_1 z_{12} + \alpha_2 z_{22}) \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1\left({}^{CF}D_{a+}^{\alpha} z_{11}\right) + \alpha_2\left({}^{CF}D_{a+}^{\alpha} z_{21}\right) \\ \alpha_1\left({}^{CF}D_{a+}^{\alpha} z_{12}\right) + \alpha_2\left({}^{CF}D_{a+}^{\alpha} z_{22}\right) \end{bmatrix} \\ &= \alpha_1 \begin{bmatrix} {}^{CF}D_{a+}^{\alpha} z_{11} \\ {}^{CF}D_{a+}^{\alpha} z_{12} \end{bmatrix} + \alpha_2 \begin{bmatrix} {}^{CF}D_{a+}^{\alpha} z_{21} \\ {}^{CF}D_{a+}^{\alpha} z_{22} \end{bmatrix} \\ &= \alpha_1 L_r z_1 + \alpha_2 L_r z_2. \end{aligned}$$

For boundedness, we need to show that $\|L_{jr}z_{jr}\|_{H_2^1}^2 \leqslant M_{jr}\|z_{jr}\|_{H_2^2}^2$ for $M_{jr} > 0$. So that $\|L_{jr}z_{jr}\|_{H_2^1}^2 = \langle L_{jr}z_{jr}(x), L_{jr}z_{jr}(x)\rangle_{H_2^1} = \int_a^b \left[(L_{jr}z_{jr})'(x)\right]^2 + [(L_{jr}z_{jr})(x)]^2 dx$. By the reproducing property of $N_x(s)$, we have $z_{jr}(x) = \langle z_{jr}(s), N_x(s)\rangle_{H_2^2}, L_{jr}z_{jr}(x) = {}^{CF}D_{x_{0+}}^{\alpha} z_{jr}(x) = \left\langle z_{jr}(s), {}^{CF}D_{x_{0+}}^{\alpha} N_x(s)\right\rangle_{H_2^2}$, $(L_{jr}z_{jr})'(x) = \langle z_{jr}(s), (L_{jr}N_{jr})'(s)\rangle_{H_2^2}$. Again, by the Schwartz Inequality, we get

$$| (L_{jr}z_{jr})(x) | = | {}^{CF}D_{x_{0+}}^{\alpha} z_{jr}(x) | = | \left\langle z_{jr}(x), {}^{CF}D_{x_{0+}}^{\alpha} N_x(s)\right\rangle_{H_2^2}$$
$$\leqslant \|{}^{CF}D_{x_{0+}}^{\alpha} G_x\|_{H_2^2}\|z_{jr}\|_{H_2^2} = M_{jr}^1\|z_{jr}\|_{H_2^2}, | (L_{jr}z_{jr})'(x) | = | \frac{d}{dx}\left({}^{CF}D_{x_{0+}}^{\alpha} z_{jr}(x)\right) | =$$
$$| \langle z_{jr}(x), \frac{d}{dx}\left({}^{CF}D_{x_{0+}}^{\alpha} N_x(s)\right)\rangle_{H_2^2} |$$
$$\leqslant \| \frac{d}{dx} {}^{CF}D_{x_{0+}}^{\alpha} N_x\|_{H_2^2}\|z_{jr}\|_{H_2^2}$$
$$= M_{jr}^2\|z_{jr}\|_{H_2^2},$$

where $M_{jr}^1, M_{jr}^2 > 0$. Thus $\|L_{jr}y_{jr}\|_{H_2^1}^2 \leqslant \left(\left(M_{jr}^1\right)^2 + \left(M_{jr}^2\right)^2\right)(b-a)\|y_{jr}\|_{H_2^1}^2$ or $\|L_{jr}y_{jr}\|_{H_2^1}^2 \leqslant M_{jr}\|z_{jr}\|_{H_2^1}^2$, in which $M_{jr} = \sqrt{\left(\left(M_{jr}^1\right)^2 + \left(M_{jr}^2\right)^2\right)(b-a)}$.

Consequently, we have $\|L_r z\|_{H^1}^2 = \sqrt{\|z_{1r}\|_{H_2^1}^2 + \|z_{2r}\|_{H_2^1}^2} \leqslant \sqrt{M_{1r} + M_{2r}}$. □

Now, the orthogonal basis of $H^2[a,b]$can be constructed as follows: let $\phi_{ij}(x) = R_{x_i}(x)e_j$ and $\psi_{ij}(x) = L_r^*\phi_{ij}(x)$ for $j = 1,2$, in which $L_r^* = diag(L_{1r}^*, L_{2r}^*), e_1 = (1,0)^\top, e_2 = (0,1)^\top, R_x(s)$ is reproducing kernel function for the space $H_2^1[a,b]$, and $\{x_i\}_{i=1}^\infty$is a dense set on $[a,b]$.

Theorem 4.2. *For IVP (4.1) and (4.2), if $\{x_i\}_{i=1}^\infty$ is dense on $[a,b]$then $\left\{\psi_{ij}(x)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$ is complete function system to $H^2[a,b]$ and $\psi_{ij}(x) = L_{rs}R_x(s)|_{s=x_i}$.*

Proof. Using the reproducing property, we have

$$\begin{aligned}\psi_{ij}(x) &= L_r^*\varphi_{ij}(x)\\ &= \langle L_r^*\varphi_{ij}(s), R_x(s)\rangle_{H^2} = \langle \varphi_{ij}(s), L_{rs}R_x(s)\rangle_{H^1} = L_{rs}R_x(s)|_{s=x_i} \in H^2[a,b].\end{aligned}$$

Moreover, for each fixed $z_r(x) \in H^2[a,b]$, let $\langle z_r(x), \psi_{ij}(x)\rangle_{H^2} = 0$ so that

$$\langle z_r(x), \psi_{ij}(x)\rangle_H = \langle z_r(x), L_r^*\psi_{ij}(x)\rangle_H \langle L_r z_r(x), \varphi_{ij}(x)\rangle_H = L_r z_r(x_i) = 0.$$

Thus $L_r z_r(x) = 0$ because $\{x_i\}_{i=1}^\infty$ is dense on $[a,b]$. Also, since L_r^{-1} exists, we have $z_r(x) = 0$.As a result, $\{\psi_{ij}(x)\}_{(i,j)=(1,1)}^{(\infty,2)}$ is complete function system for $H[a,b]$. □

Next, the Gram-Schmidt orthogonalization process is applied on $\{\psi_{ij}(x)\}_{(i,j)=(1,1)}^{(\infty,2)}$ to construct an orthonormal function system $\{\overline{\psi_{ij}}(x)\}_{(i,j)=(1,1)}^{(\infty,2)}$ of $H^2[a,b]$ as follows

$$\overline{\psi_{ij}}(x) = \sum_{l=1}^{i}\beta_{jil}\psi_{ij}(x), \quad i = 1,2,3,\ldots, \quad j = 1,2, \tag{4.3}$$

in which β_{jil} are computed by the following formulas:

$$\begin{aligned}\beta_{j11} &= {}^{1}\|\psi_{ij}\|_{H_2^2},\\ \beta_{jii} &= \frac{1}{\sqrt{\|\psi_{ij}\|_{H_2^2}^2 - \sum_{q=1}^{i-1}\langle\psi_{ij}(x),\psi_{qj}(x)\rangle_{H_2^2}^2}}, \quad if\ i = l,\\ \beta_{jil} &= \frac{um_{q=1}^{i-1}\langle\psi_{ij}(x),\psi_{qj}(x)\rangle_{H_2^2}^2\beta_{jql}}{\sqrt{\|\psi_{ij}\|_{H_2^2}^2 - \sum_{q=1}^{i-1}\langle\psi_{ij}(x),\psi_{qj}(x)\rangle_{H_2^2}^2}}, \quad if\ i > l,\ j = 1,2.\end{aligned}$$

Theorem 4.3. *Let the solution of IVP (4.1) and (4.2) be unique and $\{x_i\}_{i=1}^\infty$ be dense set on $[a,b]$, then the analytic solution has the form,*

$$z_r(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}G(x_l, z_r(x_l))\overline{\psi_{ij}}(x). \tag{4.4}$$

Proof. Let $\{x_i\}^\infty$be dense on $[a,b]$. Then, $\{\overline{\psi_{ij}}(x)\}_{(i,j)=(1,1)}^{(\infty,2)}$ is a complete orthonormal system of $H^2[a,b]$, which means that each $z_r(x) \in [a,b]$can be expanded in the Fourier series as:

$$\begin{aligned}z_r(x) &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\langle z_r(x), \overline{\psi_{ij}}(x)\rangle_{H^2}\overline{\psi_{ij}}(x)\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\langle z_r(x), \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}\psi_{lk}(x)\rangle_{H^2}\overline{\psi_{ij}}(x)\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}\langle z_r(x), L_r^*(x)\varphi_{lk}(x)\rangle_{H^2}\overline{\psi_{ij}}(x)\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}\langle L_r z_r(x), \varphi_{lk}(x)\rangle_{H^1}\overline{\psi_{ij}}(x)\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}\langle G_r(x, z_r(x)), \varphi_{lk}(x)\rangle_{H^1}\overline{\psi_{ij}}(x)\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}G_{kr}(x_l, z_r(x_l))\overline{\psi_{ij}}(x). \qquad \square\end{aligned}$$

To obtain a numerical approximation to the solution of the fractional IVP (4.1) and (4.2), we may use the finite sum:

$$z_r^n(x) = \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}G_{kr}(x_l, z_r(x_l))\overline{\psi_{ij}}(x). \tag{4.5}$$

Consequently, the n-term approximate solution $y_r^n(x)$of (2.4) is

$$y_r^n(x) = z_r^n(x) + \beta_r. \tag{4.6}$$

Algorithm 4.4. Let δ_{ijr} be defined as $\delta_{ijr} = \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}G_{kr}(x_l, z_r(x_l))$, then the solution (4.4) of IVP (4.1) and (4.2) can be written as $z_r(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\delta_{ijr}\overline{\psi}_{ij}$.

Pick the initial value $x_1 = a$ and use the homogeneous IC $z_r(a) = 0$to obtain the value of $z_r(x_1)$and consequently the value of $G_r(x_1, z_r(x_1))$. Thus, put $z_r^0(x_1) = z_r(x_1)$and define the n-term truncated solution of $z_r(x)$by

$$z_r^n(x) = \sum_{i=1}^{n}\sum_{j=1}^{2}\delta_{ijr}\overline{\psi_{ij}}(x), \tag{4.7}$$

where $\delta_{ijr}, i = 1,2,\ldots,n, j = 1,2$, can be computed using the formulas:

$$\begin{aligned}\delta_{1jr} &= \sum_{i=1}^{1}\sum_{k=1}^{j}\beta_{k1l}G_{kr}(x_l, z_r^{l-1}(x_l));\\ z_r^1(x) &= \sum_{i=1}^{1}\sum_{j=1}^{2}\delta_{ijr}\overline{\psi_{ij}}(x),\\ \delta_{2jr} &= \sum_{i=1}^{2}\sum_{k=1}^{j}\beta_{k2l}G_{kr}(x_l, z_r^{l-1}(x_l));\\ z_r^2(x) &= \sum_{i=1}^{2}\sum_{j=1}^{2}\delta_{ijr}\overline{\psi_{ij}}(x),\\ &\vdots\\ \delta_{njr} &= \sum_{i=1}^{n}\sum_{k=1}^{j}\beta_{knl}G_{kr}(x_l, z_r^{l-1}(x_l));\\ z_r^n(x) &= \sum_{i=1}^{n}\sum_{j=1}^{2}\delta_{ijr}\overline{\psi_{ij}}(x).\end{aligned}$$

5. convergence and stability analysis

This section aims to examine the convergence and stability of RKHS method for the solution of

$$L_r z_r(x) = G(x, z_r(x)), \tag{5.1}$$

where the operator $L_r$is given by Eq. (4.1).

Lemma 5.1. [44] *Assume that* $\|z_{r^{(n-1)}} - z_r\|_{H^2} \to 0$ *and* $x_n \to \tau$ *as* $n \to \infty$, $\|z_{r^{n-1}}\|_{H^2}$ *is bounded, and* $G_r(t, z_r(x))$ *is continuous. Then* $G_r(x_n, z_{r^{(n-1)}}(x_n))$*converges to* $G_r(\tau, z_r(\tau))$.

Theorem 5.2. *Assume that* $\|z_r^n\|_{H^2}$*in formula (4.7) is bounded and the solution of FFDE (2.4) is unique. Let* $\{x_i\}_{i=1}^{\infty}$ *be dense subset of* $[a, b]$*, then the n-term approximation* $z_r^n(x)$ *converges to the analytic solution* $z_r(x)$*of system (4.1) and (4.2) such that*

$$z_r(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\delta_{ijr}\overline{\psi_{ij}}(x),$$

where $\delta_{ijr} = \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}G_{kr}(x_l, z_r(x_l))\overline{\psi_{ij}}(x)$.

Proof. First, we show that $z_r^n(x)$ converges to $z_r(x)$. From (4.7), we can write $z_r^{n+1}(x) = z_r^n(x) + \sum_{j=1}^{2}\delta_{(n+1)jr}\overline{\psi_{(n+1)j}}(x)$. From the orthonormality of $\{\overline{\psi_{ij}}(x)\}_{(i,j)=(1,1)}^{(\infty,2)}$,it follows that

$$\begin{aligned}\|z_r^{n+1}\|_{H^2}^2 &= \|z_r^n\|_{H^2}^2 + \sum_{j=1}^{2}(B_{(n+1)jr})^2 \\ &= \|z_r^{n-1}\|_{H^2}^2 + \sum_{j=1}^{2}(B_{(n)jr})^2 + \sum_{j=1}^{2}(B_{(n+1)jr})^2 = \cdots = \|z_r^0\|_{H^2}^2 + \sum_{i=1}^{n+1}\sum_{j=1}^{2}(\delta_{ijr})^2.\end{aligned}$$

Hence, it holds that $\|z_r^{n+1}\|_{H^2}^2 \geqslant \|z_r^n\|_{H^2}^2$. Since $\|z_r^n\|_H^2$ is bounded, then its convergent and hence $\exists c \in \mathrm{R}$ so that $\sum_{i=1}^{\infty}\sum_{j=1}^{2}(\delta_{ijr})^2 = c$. Thus, $\left\{\sum_{j=1}^{2}(\delta_{ijr})^2\right\}_{i=1}^{\infty} \in l^2$. However, since $(z_r^\eta(x) - z_r^{\eta-1}(x)) \perp (z_r^{\eta-1}(x) - z_r^{\eta-2}(x)) \perp \cdots \perp (z_r^{n+1}(x) - z_r^n(x))$, it follows for $\eta > n$that $\|z_{r^\eta} - z_{r^n}\|_{H^2}^2 = \|z_{r^\eta} - z_{r^{m-1}} + \cdots + z_{r^{n+1}} - z_{r^n}\|_{H^2}^2 = \|z_{r^\eta} - z_{r^{\eta-1}}\|_{H^2}^2 + \|z_{r^{\eta-1}} - z_{r^{\eta-2}}\|_{H^2}^2 + \cdots + \|z_{r^{n+1}} - z_{r^n}\|_{H^2}^2$.

Also, $\|z_r^\eta - z_r^{\eta-1}\|_{H^2} = \sum_{j=1}^{2}(\delta_{\eta jr})^2$. Thus, as $n, \eta \to \infty$, we get $\|z_{r^\eta} - z_{r^{n-1}}\|_{H^2} = \sum_{l=n+1}^{\eta}\left(\sum_{j=1}^{2}(\delta_{ljr})^2\right) \to 0$. With the help of completeness of $H^{2[a,b]}, \exists z_r(x) \in H[a, b]$ such that $z_r^n(x) \to z_r(x)$ as $n \to \infty$, i.e., $z_{r^n}(x)$ is convergent.

To complete the proof, we have to show that $z_r(x)$is a solution of (2.4). Taking limits on both sides of Eq. (4.7), we get $z_r(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\delta_{ijr}\overline{\psi_{ij}}(x)$. While $(L_r z_r)(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\delta_{ijr}L_r\overline{\psi_{ij}}(x)$, so

$$\begin{aligned}(L_r z_r)_k(x_l) &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\delta_{ijr}\langle L_r\overline{\psi_{ij}}(x), \varphi_{ij}(x)\rangle_{H^1} \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\delta_{ijr}\langle\overline{\psi_{ij}}(x), L_{r^*}\varphi_{ij}(x)\rangle_{H^2} \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\delta_{ijr}\langle\overline{\psi_{ij}}(x), \psi_{lk}(x)\rangle_{H^2}.\end{aligned}$$

After simple calculations, we get

$$\begin{aligned}\sum_{l'=1}^{l}\sum_{k'=1}^{k}\beta_{l'k'}^{lk}(L_r z_r)_{k'}(x_{l'}) &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\delta_{ijr}\langle\overline{\psi_{ij}}(x), \sum_{l'=1}^{l}\sum_{k'=1}^{k}\beta_{l'k'}^{lk}\psi_{l'k'}(x)\rangle_{H^2} \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\delta_{ijr}\langle\overline{\psi_{ij}}(x), \overline{\psi_{l'k'}}(x)\rangle_{H^2} = \delta_{lkr}.\end{aligned}$$

For $l = 1, (L_r z_r)_j(x_1) = G_{jr}(x_1, z_r^0(x_1)), j = 1, 2$. Which means $(L_r z_r)(x_1) = G_r(x_1, z_{r^0}(x_1))$.

For $l = 2, (L_r z_r)_j(x_2) = G_{jr}(x_2, z_r^1(x_2)), j = 1, 2$. Which means $(L_r z_r)(x_2) = G_r(x_l, z_{r^1}(x_2))$.

Continuing this manner, we can discover that $(L_r z_r)(x_n) = G_r(x_n, z_{r^{n-1}}(x_n))$. Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[a, b]$, then for each $s \in [a, b]$, there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that $x_{n_j} \to \infty$. But $L_r z_r(x_{n_j}) = G_r(x_{n_j}, z_{r^{n_j-1}}(x_{n_j}))$. By Lemma (5.1) and continuity of G_r, we have $L_r z_r(s) = G_r(x, z_r(s))$. Since $\overline{\psi_{ij}}(x) \in H^{2[a,b]}$ is unique, then $z_r(x)$ satisfy (2.4). i.e., $z_r(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\delta_{ijr}\overline{\psi_{ij}}(x)$. □

Lemma 5.3. [45] *If* $z(x) \in H_2^2[a, b]$ *and* $z^{(\eta)}, \eta = 0, 1$*, denotes the* η^{th} *derivative of z, then there is a constant* $\xi > 0$ *that satisfies*

$$\|z^{(\eta)}\|_\infty \leqslant \xi\|z\|_{H_2^2[a,b]}, \tag{5.2}$$

where $\|z\|_\infty = sup\{|z(x)|, x \in [a, b]\}$.

Theorem 5.4. *If the fuzzy fractional IVP (5.1) has a unique solution, and* $L_r : H^2[0, 1] \longrightarrow H^1[0, 1]$*is linear and bounded. Then, the RKHS solution is stable.*

Proof. Let $z_r^n(x)$ be the RKHS-approximate solution in the form

$$z_r^n(x) = \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}G_{kr}(x_l, z_r(x_l))\overline{\psi_{ij}}(x),$$

where $x_l \in [a, b], \overline{\psi_{ij}}(x)$ and β_{kil} are presented in the previous section. Now, let $\Upsilon_r(x)$ be the approximate solution of

$$L_r\Upsilon_r(x) = G(x, \Upsilon_r(x)) + \varepsilon_r(x),$$

where $\varepsilon_r(x)$is a small non-negative bounded perturbation. Herein, we will prove that there exists constant $\sigma > 0$ such that $\|\Upsilon_r^n - z_r^n\|_\infty < \sigma$. Due to the definition of $z_r^n(x)$ and $\Upsilon_r^n(x)$, we have

$$\begin{aligned}\Upsilon_r^n(x) - z_r^n(x) &= \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}(G(x_l, \Upsilon_r(x_l)) + \varepsilon_r(x_l))\overline{\psi_{ij}(x)} \\ &-\sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}G_{kr}(x_l, z_r(x_l))\overline{\psi_{ij}}(x) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}\varepsilon_r(x_l)\overline{\psi}_{ij}(x).\end{aligned} \tag{5.3}$$

On the other aspect as well, L_r^{-1}exists and $L_r^{-1}\varepsilon_r(x) \in H^2[a, b]$. Therefore, we get

$$\begin{aligned}L_r^{-1}\varepsilon_r(x) &= \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}\langle L_r^{-1}\varepsilon_r(x), \psi_{lk}(x)\rangle_{H^2}\overline{\psi_{ij}}(x) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}\langle\varepsilon_r(x), (L_r^{-1})^*\psi_{lk}(x)\rangle_{H^1}\overline{\psi_{ij}}(x) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}\langle\varepsilon_r(x), \psi_{lk}(x)\rangle_{H^1}\overline{\psi_{ij}}(x) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{kil}\varepsilon_{kr}(x_l)\overline{\psi_{ij}}(x).\end{aligned} \tag{5.4}$$

Consequently,

$\Upsilon_r^n(x) - z_r^n(x) = L_r^{-1}\varepsilon_r(x)$.

Since $L_r^{-1}$is continuous on $[a,b]$and bounded, then we have $\|\Upsilon_r^n(x) - z_r^n(x)\|_{H^2} = \|L_r^{-1}\|\|\varepsilon_r(x)\|$. Hence, we conclude that $\|\Upsilon_r^n(x) - z_r^n(x)\|_{H^2} = M\|\varepsilon_r(x)\|$, where $M = \|L_r^{-1}\|$. Based on Lemma 5.1, $\|\Upsilon_r^n(x) - z_r^n(x)\|_\infty \leqslant cM\varepsilon$ and therefore $\sigma = kM\varepsilon$. □

6. Numerical applications

In this section, the RK algorithm that has been discussed in Section 4 will be applied to solve fuzzy fractional differential equations in Caputo-Fabrizio generalized H-differentiability. We give nonlinear examples along with a comparison between the classical Caputo fractional derivative and CF-derivative with non-singular kernel. The comparison is given for different values of r and different values of the fractional order α. The second example is an important application of mathematical model arising in biology and medicine. Specifically, a well-known equation results from pharmacokinetic models are discussed in Hilbert spaces.

Example 6.1. In preparation for numerical investigation and exploration of the fractional fuzzy model for drug pharmacokinetics, we present a brief description of this clinical model, which emerges as a natural model about rates of drug

Table 1 Numerical result of Example 6.1, Case 1 at $x = 1$ and $\alpha = 0.99$.

r	Exact $y_{1r}(x)$	Exact $y_{2r}(x)$	RK-solution $y_{1r}(x)$	RK-solution $y_{2r}(x)$	Absolute Error $y_{1r}(x)$	Absolute Error $y_{2r}(x)$
			$k_1 = 0.6931$			
0.0	−0.492209	1.462580	−0.491249	1.463480	9.60124×10^{-4}	8.93160×10^{-4}
0.1	−0.394470	1.364840	−0.393513	1.365740	9.56776×10^{-4}	8.96508×10^{-4}
0.2	−0.296730	1.267100	−0.295777	1.268000	9.53427×10^{-4}	8.99856×10^{-4}
0.3	−0.198991	1.169360	−0.195777	1.170270	3.21386×10^{-3}	9.03204×10^{-4}
0.4	−0.101251	1.071620	−0.100304	1.072530	9.46731×10^{-4}	9.06552×10^{-4}
0.5	−0.003511	0.973884	−0.002568	0.974794	9.43383×10^{-4}	9.09901×10^{-4}
0.6	0.094228	0.876145	0.095168	0.877058	9.40035×10^{-4}	9.13249×10^{-4}
0.7	0.191968	0.778405	0.192904	0.779322	9.36686×10^{-4}	9.16597×10^{-4}
0.8	0.289707	0.680666	0.290641	0.681586	9.33338×10^{-4}	9.19945×10^{-4}
0.9	0.387447	0.582926	0.388377	0.583849	9.29999×10^{-4}	9.23293×10^{-4}
1.0	0.485186	0.485186	0.486113	0.486113	9.26642×10^{-4}	9.26642×10^{-4}
			$k_1 = 0.11$			
r	Exact $y_{1r}(x)$	Exact $y_{2r}(x)$	RK-solution $y_{1r}(x)$	RK-solution $y_{2r}(x)$	Absolute Error $y_{1r}(x)$	Absolute Error $y_{2r}(x)$
0.0	−0.875600	1.079190	−0.875409	1.079320	1.90688×10^{-4}	1.23723×10^{-4}
0.1	−0.777860	0.981452	−0.777673	0.981579	1.87339×10^{-4}	1.27072×10^{-4}
0.2	−0.680120	0.883713	−0.679936	0.883843	1.83991×10^{-4}	1.30420×10^{-4}
0.3	−0.582381	0.785973	−0.579936	0.786107	2.44442×10^{-3}	1.33768×10^{-4}
0.4	−0.484641	0.688234	−0.484464	0.688371	1.77295×10^{-4}	1.37116×10^{-4}
0.5	−0.386902	0.590494	−0.386728	0.590634	1.73947×10^{-4}	1.40464×10^{-4}
0.6	−0.289162	0.492754	−0.288992	0.492898	1.70598×10^{-4}	1.43813×10^{-4}
0.7	−0.191423	0.395015	−0.191255	0.395162	1.67250×10^{-4}	1.47161×10^{-4}
0.8	−0.093683	0.297275	−0.093519	0.297426	1.63902×10^{-4}	1.50509×10^{-4}
0.9	0.004057	0.199536	0.004217	0.199690	1.60554×10^{-4}	1.53857×10^{-4}
1.0	0.101796	0.101796	0.101953	0.101953	1.57206×10^{-4}	1.57206×10^{-4}
			$k_1 = 0.3$			
r	Exact $y_{1r}(x)$	Exact $y_{2r}(x)$	RK-solution $y_{1r}(x)$	RK-solution $y_{2r}(x)$	Absolute Error $y_{1r}(x)$	Absolute Error $y_{2r}(x)$
0.0	−0.724688	1.230100	−0.724236	1.230490	4.52475×10^{-4}	3.85511×10^{-4}
0.1	−0.626949	1.132360	−0.626500	1.132750	4.49127×10^{-4}	3.88859×10^{-4}
0.2	−0.529209	1.034620	−0.528763	1.035020	4.45779×10^{-4}	3.92207×10^{-4}
0.3	−0.431470	0.936884	−0.428763	0.937280	2.70621×10^{-3}	3.95556×10^{-4}
0.4	−0.333730	0.839145	−0.333291	0.839544	4.39082×10^{-4}	3.98904×10^{-4}
0.5	−0.235990	0.741405	−0.235555	0.741808	4.35734×10^{-4}	4.02252×10^{-4}
0.6	−0.138251	0.643666	−0.137818	0.644071	4.32386×10^{-4}	4.05600×10^{-4}
0.7	−0.040511	0.545926	−0.040082	0.546335	4.29038×10^{-4}	4.08948×10^{-4}
0.8	0.057228	0.448187	0.057654	0.448599	4.25690×10^{-4}	4.12297×10^{-4}
0.9	0.154968	0.350447	0.155390	0.350863	4.22341×10^{-4}	4.15645×10^{-4}
1.0	0.252707	0.252707	0.253126	0.253126	4.18993×10^{-4}	4.18993×10^{-4}

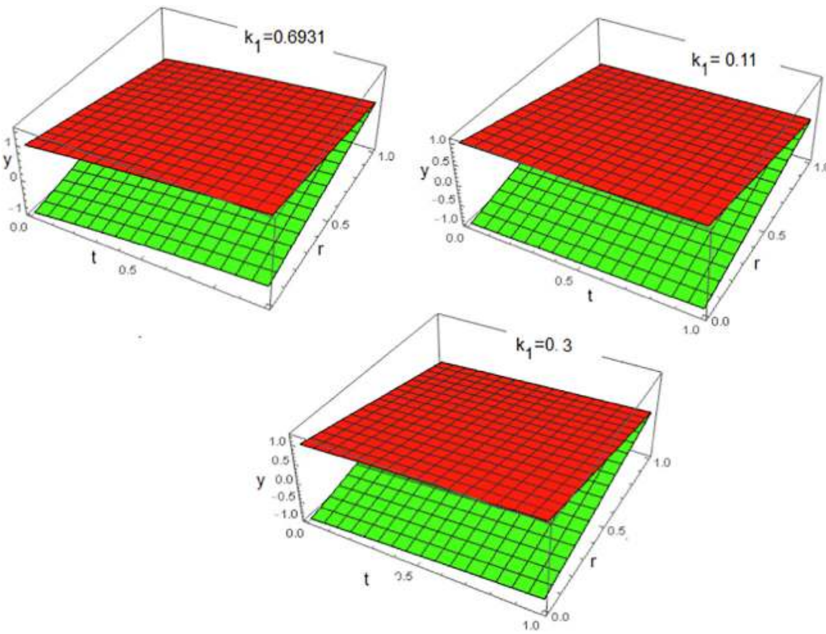

Fig. 1 3D plots of fuzzy approximate solution at $\alpha = 0.99$ under ${}^{CF}[1-\alpha]$- differentiability for Example 6.1, Case 1.

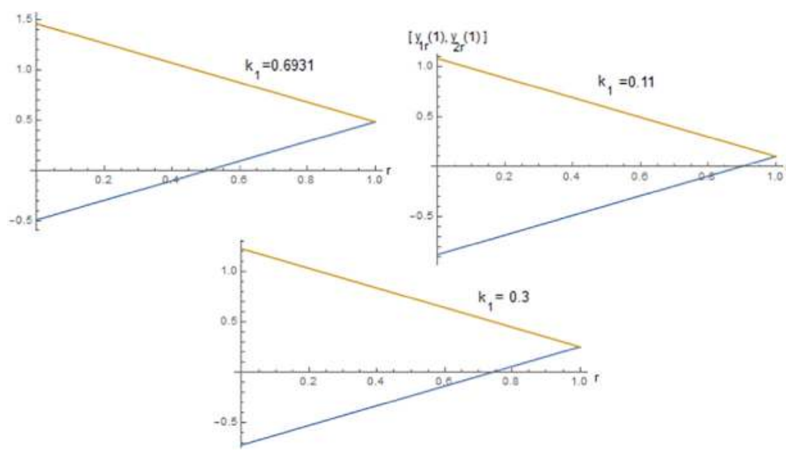

Fig. 2 The level RK-fuzzy approximate solutions at $x = 1$ and $\alpha = 0.99$ under ${}^{CF}[1-\alpha]$- differentiability of Example 6.1, Case 1.

absorption into the bloodstream under potential uncertainty. Usually, pharmacokinetic pills are taken readily without the patient necessarily having an adequate understanding of the way in which these drugs are absorbed into the bloodstream or the extent and speed of the effective effect on the body. This pharmacokinetic model reveals very different levels of absorption and extraction rates of drugs of all kinds from the bloodstream depending on multiple factors, including bioavailability, active ingredients, its physical and chemical properties, physiological characteristics of the patient taking the drug, etc.

Before we give the fractional fuzzy DE of the pharmacokinetics model and study it numerically, we give a brief description for such medical model. In fact, the pharmacokinetics model results as a natural phenomenon under potential uncertainties. It is related to the drug assimilation into the blood. We readily take pills without necessarily having a good understanding of how these drugs are absorbed into the bloodstream or for how long they have an effect on us. This model reveals that different drugs are absorbed into and extracted from the blood at very different rates. In this context, absorption of a drug represents the pharmacokinetics of the bloodstream, whereby the drug is initially dissolved in the gastrointestinal tract (GI-tract) and then its components diffuse into the blood, while the active ingredients are carried to the desired locations and then removed from the blood by the kidneys and liver. Diet quality and digestive disorders can also affect drug absorption and bioavailability. Thus, this

Table 2 Numerical results for the RK-solution of Example 6.1, Case 2 at $x = 1$ and $\alpha = 0.99$.

r	Exact $y_{1r}(x)$	Exact $y_{2r}(x)$	RK-solution $y_{1r}(x)$	RK-solution $y_{2r}(x)$	Absolute Error $y_{1r}(x)$	Absolute Error $y_{2r}(x)$
			$k_2 = 0.01386$			
0.0	−0.260557	1.712190	−0.258804	1.713900	1.75300×10^{-3}	1.71277×10^{-3}
0.1	−0.161919	1.613550	−0.160168	1.615270	1.75099×10^{-3}	1.71478×10^{-3}
0.2	-0.063282	1.514920	−0.061533	1.516630	1.74898×10^{-3}	1.71679×10^{-3}
0.3	0.035355	1.416280	0.038467	1.418000	3.11156×10^{-3}	1.71880×10^{-3}
0.4	0.133993	1.317640	0.135738	1.319360	1.74496×10^{-3}	1.72081×10^{-3}
0.5	0.232630	1.219000	0.234373	1.220730	1.74294×10^{-3}	1.72282×10^{-3}
0.6	0.331268	1.120370	0.333009	1.122090	1.74093×10^{-3}	1.72484×10^{-3}
0.7	0.429905	1.021730	0.431644	1.023460	1.73892×10^{-3}	1.72685×10^{-3}
0.8	0.528543	0.923092	0.530280	0.924821	1.73691×10^{-3}	1.72886×10^{-3}
0.9	0.627180	0.824455	0.628915	0.826186	1.73490×10^{-3}	1.73087×10^{-3}
1.0	0.725817	0.725817	0.727550	0.727550	1.73288×10^{-3}	1.73288×10^{-3}
			$k_2 = 0.06386$			
r	Exact $y_{1r}(x)$	Exact $y_{2r}(x)$	RK-solution $y_{1r}(x)$	RK-solution $y_{2r}(x)$	Absolute Error $y_{1r}(x)$	Absolute Error $y_{2r}(x)$
0.0	−0.234846	1.642700	−0.233057	1.644310	1.78891×10^{-3}	1.60525×10^{-3}
0.1	−0.140969	1.548820	−0.139189	1.550440	1.77973×10^{-3}	1.61443×10^{-3}
0.2	−0.047092	1.454950	−0.045321	1.456570	1.77054×10^{-3}	1.62361×10^{-3}
0.3	0.046786	1.361070	0.054679	1.362700	7.89319×10^{-3}	1.63280×10^{-3}
0.4	0.140663	1.267190	0.142415	1.268830	1.75218×10^{-3}	1.64198×10^{-3}
0.5	0.234540	1.173310	0.236283	1.174970	1.74299×10^{-3}	1.65116×10^{-3}
0.6	0.328418	1.079440	0.330152	1.081100	1.73381×10^{-3}	1.66035×10^{-3}
0.7	0.422295	0.985559	0.424020	0.987229	1.72463×10^{-3}	1.66953×10^{-3}
0.8	0.516172	0.891682	0.517888	0.893361	1.71544×10^{-3}	1.67871×10^{-3}
0.9	0.610050	0.797804	0.611756	0.799492	1.70626×10^{-3}	1.68789×10^{-3}
1.0	0.703927	0.703927	0.705624	0.705624	1.69708×10^{-3}	1.69708×10^{-3}
			$k_2 = 0.1386$			
r	Exact $y_{1r}(x)$	Exact $y_{2r}(x)$	RK-solution $y_{1r}(x)$	RK-solution $y_{2r}(x)$	Absolute Error $y_{1r}(x)$	Absolute Error $y_{2r}(x)$
0.0	−0.199246	1.544650	−0.197414	1.546090	1.83255×10^{-3}	1.44193×10^{-3}
0.1	−0.112051	1.457460	−0.110238	1.458920	1.81302×10^{-3}	1.46146×10^{-3}
0.2	−0.024857	1.370260	−0.023063	1.371740	1.79349×10^{-3}	1.48099×10^{-3}
0.3	0.062338	1.283070	0.076937	1.284570	1.45986×10^{-2}	1.50053×10^{-3}
0.4	0.149533	1.195870	0.151288	1.197390	1.75442×10^{-3}	1.52006×10^{-3}
0.5	0.236728	1.108680	0.238463	1.110220	1.73489×10^{-3}	1.53959×10^{-3}
0.6	0.323923	1.021480	0.325639	1.023040	1.71536×10^{-3}	1.55912×10^{-3}
0.7	0.411118	0.934288	0.412814	0.935866	1.69583×10^{-3}	1.57865×10^{-3}
0.8	0.498313	0.847093	0.499989	0.848691	1.67630×10^{-3}	1.59818×10^{-3}
0.9	0.585508	0.759898	0.587165	0.761516	1.65677×10^{-3}	1.61771×10^{-3}
1.0	0.672703	0.672703	0.674340	0.674340	1.63724×10^{-3}	1.63724×10^{-3}

overlapping problem can be studied as a segmented mathematical model of two parts comprising the GI-tract and bloodstream having single inputs and outputs [48].

Consequently, let us assume that $z(x)$ and $y(x)$ represent the amount of drug in the GI-tract and in the bloodstream at time x, respectively. Anyhow, we assume that the pill must be swallowed to reach the GI-tract, without entering anything else during the subsequent period. This pill dissolves in the GI-tract and then diffuses into the bloodstream. Therefore, the GI-tract can be considered only an output source. Taking into account that the output rate is actually proportional to the drug concentration in the GI-tract and to the drug amount in the bloodstream [49]. To this end, consider the following linear differential equation:

$$\frac{dz}{dx} = -k_1 z, \;\; z(0) = z_0, \tag{6.1}$$

where z_0 is the drug amount in the pill and k_1 is a real positive constant. Bearing in mind that the initial drug amount in the bloodstream is zero, i.e., $y(0) = 0$. Due to the drug diffuses from the GI-tract, levels of the drug increase in the bloodstream, whereas if the drug is removed through the kidneys and liver, the drug levels gradually decrease. Therefore, IVP (6.1) becomes as follows

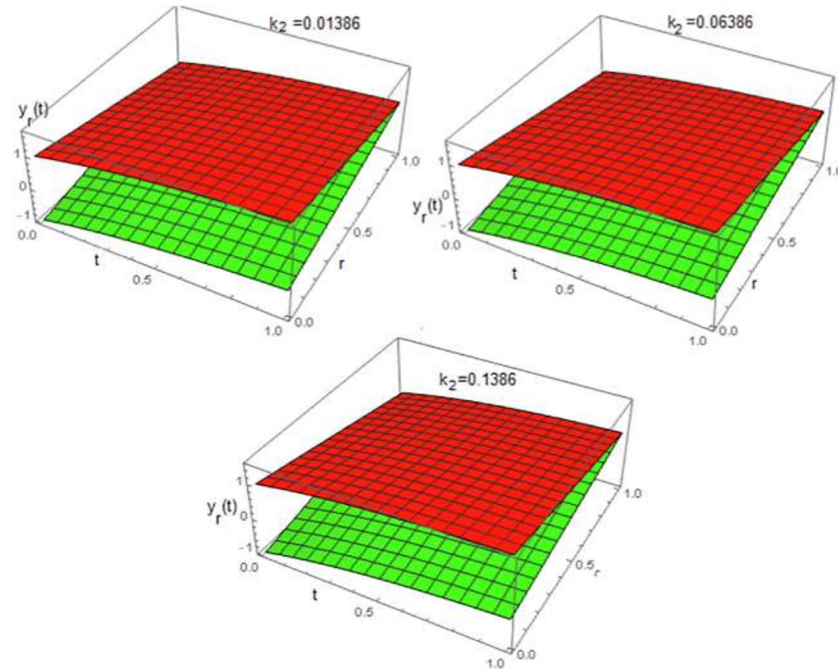

Fig. 3 3D plots of fuzzy approximate solutions at $\alpha = 0.99$ under ${}^{CF}[2-\alpha]$- differentiability for Example 6.1, Case 2.

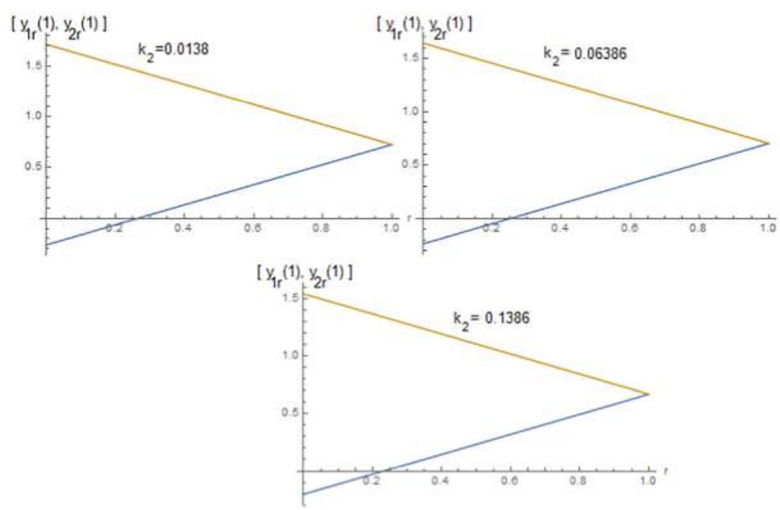

Fig. 4 The level RK-fuzzy approximate solutions at $x = 1$and $\alpha = 0.99$ under ${}^{CF}[2-\alpha]$- differentiability of Example 6.1, Case 2.

$$\frac{dy}{dx} = k_1 z - k_2 y, \;\; y(0) = 0, \tag{6.2}$$

in which $k_2 > 0$ and $\frac{dy}{dx}$ is the drug concentration. The growth and dissolution of antihistamine levels in the GI-tract and bloodstream can be discussed by replacing the z-value, $z = Ae^{-k_1 x}$, of (6.1), which is completely independent of y, in IVP (6.2) as follows,

$$\frac{dy}{dx} = k_1 A e^{-k_1 x} - k_2 y, \;\; y(0) = 0. \tag{6.3}$$

To further generalize, the drug concentration within the influence compartment is evaluated in the Caputo-Fabrizio fractional sense under certainty as follows,

$${}^{CF}D^{\alpha} y(x) + k_2 y(x) = k_1 A \exp^{-k_1 x}, \;\; 0 \leqslant \alpha \leqslant 1, \tag{6.4}$$

along with fuzzy initial condition

$$y(0, r) = [r-1, 1-r],$$

where $y(t)$is a continuous fuzzy valued function and ${}^{CF}D^{\alpha}_{0^+}$is the fuzzy Caputo-Fabrizio derivative of order α.

Assuming $y(x)$ is ${}^{CF}(1-\alpha)$-differentiable, the fuzzy fractional IVP (6.4) is equivalent to the system

$${}^{CF}D^{\alpha} y_{1r}(x) + k_2 y_{1r}(x) = k_1 A e^{-k_1 x},$$

$${}^{CF}D^{\alpha} y_{2r}(x) + k_2 y_{2r}(x) = k_1 A e^{-k_1 x},$$

Table 3 Numerical comparison of fuzzy solution at $x = 1$ of Example 6.2, Case 1.

	$y_{1r}(1)$					
	Caputo			Caputo-Fabrizio		
r	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
0.0	−0.7955043	-0.8357364	−0.8807719	−0.7962385	−0.8461976	-0.9002451
0.2	−0.7161114	-0.7533557	−0.7957156	−0.7165481	−0.7643323	−0.8168187
0.4	−0.6243834	-0.6579076	−0.6966515	−0.6245971	−0.6690880	−0.7188927
0.6	−0.5134479	-0.5420520	−0.5756330	−0.5135317	−0.5528378	−0.5979164
0.8	−0.3657848	-0.3870655	−0.4124288	−0.3658315	−0.3961392	−0.4320204

	$y_{2r}(1)$					
	Caputo			Caputo-Fabrizio		
r	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
0.0	0.7955043	0.8357364	0.8807719	0.7962385	0.8461976	0.9002451
0.2	0.7161114	0.7533557	0.7957156	0.7165481	0.7643323	0.8168187
0.4	0.6244055	0.6579076	0.6966515	0.6245971	0.6690880	0.7188927
0.6	0.5134479	0.5420520	0.5756330	0.5135317	0.5528378	0.5979164
0.8	0.3657848	0.3870655	0.4124288	0.3658315	0.3961392	0.4320204

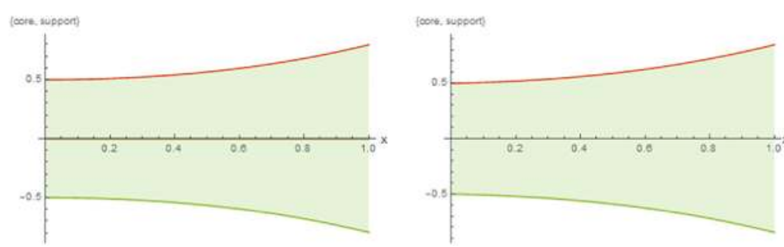

Fig. 5 The RK core and support for $\alpha = 1$ (left), and $\alpha = 0.9$ (right) under ${}^{CF}[1-\alpha]$- differentiability for Example 6.2.

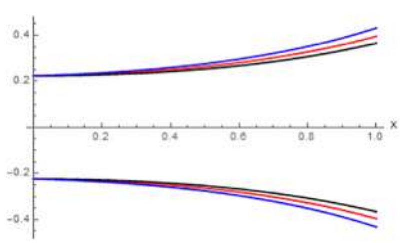

Fig. 6 The approximate solution $y_{1r}(x)$and $y_{2r}(x)$at $r = 0.8$and different values of α, Example 6.2, Case1.

$$y_{1r}(0) = r - 1,$$

$$y_{2r}(0) = 1 - r,$$

whose exact solution for arbitrary $\alpha \in (0, 1]$is

$$y_{1r}(t) = \frac{e^{-xk_1}\left(k_1\gamma - k_1(\gamma + r - 1)e^{\frac{\gamma k_2 x}{(\alpha-1)k_2-1}} + (r-1)\gamma k_2\right)}{\alpha k_2 + k_1(-1 + (-1+\alpha)k_2)},$$

$$y_{2r}(x) = \frac{e^{-xk_1}\left(k_1\gamma - k_1(\gamma + 1 - r)e^{\frac{\gamma k_2 x}{(\alpha-1)k_2-1}} + (1-r)\gamma k_2\right)}{\alpha k_2 + k_1(-1 + (-1+\alpha)k_2)},$$

where $\gamma = \alpha + (\alpha - 1)k_1$.

Using RKHS technique for $n = 30, \alpha = 0.99$and $A = 1$, the aforementioned system can be solved numerically based on the following two cases:

Case 1: Set the value of $k_2$and change k_1: let $k_2 = 0.0231$and $k_1 \in \{0.6931, 0.3, 0.11\}$, the absolute error values when r varies from 0 to 1 for different values of k_1 are presented in Table 1. Obviously, the absolute error increases as a result of the increasing in the values of k_1 which is considered as the source of variation of the drug in the bloodstream comparing to the amount of drug in the GI-tract. The values of the error, however, are proved to be proportional to the divergence of exact solutions and corresponding to the uncertainty in the approximate solutions. The RK-solution for $\alpha = 0.99$ and $k_1 \in \{0.6931, 0.3, 0.11\}$ are shown in Figs. 1 and 2.

Case 2: Set the value of $k_1$and change $k_2$with $n = 30, \alpha = 0.99$and $A = 1$so that the acquired results are presented in Table 2 and Figs. 3 and 4.

From these results, the downward trend for fixed r is remarked by increasing the values of k_2, which conflicts with the trend noticed in Table 1 for various k_1 values. Further, the values of absolute errors when r vary from 0 to 1 for different values of k_2 are listed in Table 2, whereas the parameter k_2 of $y(x)$ indicates the clearance constant for the drug amount in the bloodstream. By reducing the value of k_2, it is evident that the drug absorption from the bloodstream by the kidney and

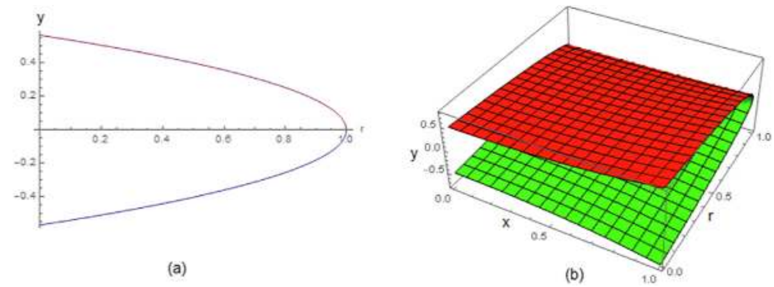

Fig. 7 The fuzzy approximate solution under ${}^{CF}[1-\alpha]$-differentiability when $x = 0.5$ for Example 6.2.

Table 4 Numerical comparison of fuzzy solution at $r = 0.5$ of Example 6.2, Case 2.

	$y_{1r}(x)$ at $r = 0.5$					
	Caputo			Caputo-Fabrizio		
t	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
0.2	−0.34655819	−0.34445014	−0.34202936	−0.34636993	−0.34052088	−0.33470856
0.4	−0.32644889	−0.32129156	−0.31563671	−0.32626722	−0.31703161	−0.30820277
0.6	−0.29562389	−0.28811784	−0.28041241	−0.29545587	−0.28578006	−0.27666401
0.8	−0.25741363	−0.24917798	−0.24127375	−0.25729989	−0.24975935	−0.24255141
1.0	−0.21514500	−0.20840197	−0.20208622	−0.21549007	−0.21184749	−0.20797548
	$y_{2r}(x)$ at $r = 0.5$					
	Caputo			Caputo-Fabrizio		
r	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
0.2	0.34655819	0.34445014	0.34202936	0.34636993	0.34052088	0.33470856
0.4	0.32644889	0.32129156	0.31563671	0.32626722	0.31703161	0.30820277
0.6	0.29562389	0.28811784	0.28041241	0.29545587	0.28578006	0.27666401
0.8	0.25741363	0.24917798	0.24127375	0.25729989	0.24975935	0.24255141
1.0	0.21514500	0.20840197	0.20208622	0.21549007	0.21184749	0.20797548

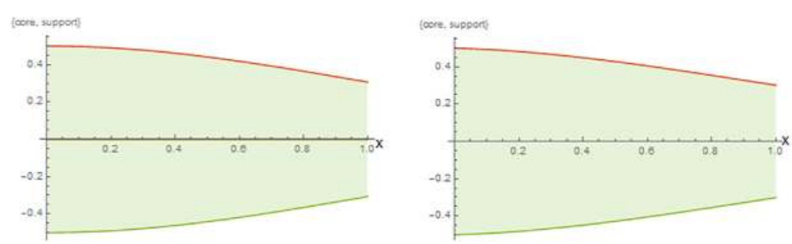

Fig. 8 The RK core and support for $\alpha = 1$ (left), and $\alpha = 0.9$ (right) under ${}^{CF}[2-\alpha]$-differentiability, Example 6.2.

liver decreases and the drug remains in the blood for a longer period, and this often occurs in the sick and elderly.

Example 6.2. Consider the following nonlinear fuzzy fractional IVP:

$$ {}^{CF}D^{\alpha}_{0^+}y(x) = \tanh(xy(x)), \quad 0 < \alpha \leqslant 1, \quad x \in [0,1], \qquad y(0) = u \in \mathbb{R}_g, \tag{6.5} $$

in which $u(s) = \max(0, 1 - 4s^2), \; s \in \mathbb{R}$.

To apply the proposed RK-algorithm, we convert the FFIVP (6.5) into the following systems of fractional IVPs:

Case1: Under ${}^{CF}[(1)-\alpha]$-differentiability, the system can be written by

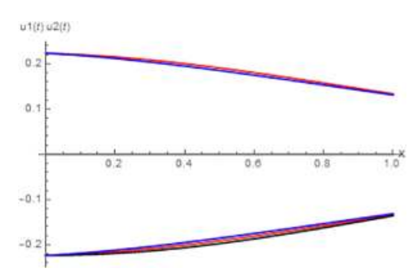

Fig. 9 The approximate solution $y_{1r}(x)$and $y_{2r}(x)$at $r = 0.8$and different values of $\alpha$of Example 6.2: Case2.

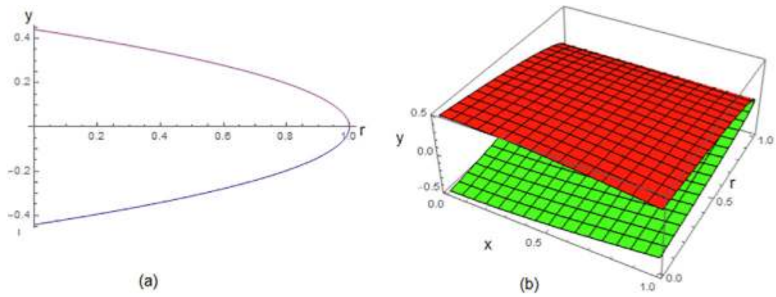

Fig. 10 Fuzzy approximate solution under ${}^{CF}[2-\alpha]$-differentiability when $x = 0.5$ for Example 6.2.

$${}^{CF}D^{\alpha}_{0^+}y_{1r}(x) = \tanh(xy_{1r}(x)),$$

$${}^{CF}D^{\alpha}_{0^+}y_{2r}(x) = \tanh(xy_{2r}(x)),$$

$$y_{1r}(0) = -1/2\sqrt{1-r},$$

$$y_{2r}(0) = 1/2\sqrt{1-r}.$$

Case2: Under ${}^{CF}[(2)-\alpha]$-differentiability, the system can be written by

$${}^{CF}D^{\alpha}_{0^+}y_{1r}(x) = \tanh(xy_{2r}(x)),$$

$${}^{CF}D^{\alpha}_{0^+}y_{2r}(x) = \tanh(xy_{1r}(x)),$$

$$y_{1r}(0) = -1/2\sqrt{1-r},$$

$$y_{2r}(0) = 1/2\sqrt{1-r}.$$

According the proposed method by taking $n = 30, x = 1$, and $r \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$, the fractional effects of the solutions of Example 6.2 using the classical Caputo and Caputo-Fabrizio operators are presented in Table 3. Moreover, Fig. 5 compares the cores and supports for $\alpha = 1$and $\alpha = 0.9$under ${}^{CF}[(1)-\alpha]$-differentiability using Caputo-Fabrizio operator. While Fig. 6 shows the curves for the RK-solution for $r = 0.8$ and different values of α. Obviously, as the fractional Caputo-Fabrizio derivative approaches the integer order derivative, the solution curve approaches the solution curve for classical integer order fuzzy IVP. The behaviour of the CF-solution with $\alpha = 0.9$ and $t = 0.5$ as r varies over $[0, 1]$ is given in Fig. 7(a), while a 3D graph is given in Fig. 7(b). Similarly, Table 4 shows a comparison between Caputo and Caputo-Fabrizio operators to the solution of the FFIVP (6.2) under ${}^{CF}[2-\alpha]$-differentiability. From these tables, it is obvious that the approximate solutions achieved by Caputo-Fabrizio sense are more accurate than the classic Caputo sense compared to the exact solution when $\alpha = 1$. The RK core and support for $\alpha = 1$ and $\alpha = 0.9$ are presented in Fig. 8. The solution curves for $\alpha \in \{1, 0.9, 0.8\}$ are given in Fig. 9. Fig. 10(a) displays the solution when x is fixed and r is variable for the CF-order $\alpha = 0.9$. While the 3D for the solution is given in Fig. 10(b).

7. Conclusion

In this paper, the RKHS method has been applied lucratively to solve first-order fuzzy fractional differential equations via a novel Caputo-Fabrizio fractional derivative of order $\alpha \in (0, 1]$, including the pharmacokinetic equation which results from the absorption of the drug into the bloodstream. The obtained results show the accuracy and high quality of the proposed method, especially in nonlinear cases. The approximate solutions have been presented graphically in closed form. Also, the RK approximate solutions have been compared by using Caputo-Fabrizio and classical Caputo concepts. Comparing the results showed the effectiveness and strength of the proposed method in solving different types of fuzzy fractional problems.In future research, this method can be applied to achieve analytical and approximate solutions of perturbed fractional differential equations under the uncertainty equipped with non-classical and integral boundary conditions in light of Caputo-Fabrizio.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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