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Original Article

Analytic solutions of the generalized water wave dynamical equations based on time-space symmetric differential operator

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Abstract

It is well known that there is a deep connection between the symmetric and traveling wave solutions. It has been shown that all symmetric waves are traveling waves. In this paper, we establish new analytic solution collections of nonlinear conformable time-fractional water wave dynamical equation in a complex domain. For this purpose, we construct a new definition of a symmetric conformable differential operator (SCDO). The operator has a symmetric representation in the open unit disk. By using SCDO, we generalize a class of water wave dynamical equation type time-space fractional complex Ginzburg–Landau equation. The results show that the obtainable approaches are powerful, dependable and prepared to apply to all classes of complex differential equations.

Keywords: Analytic solution; Conformable calculus; Fractional calculus; Water wave equations; Majorization; Subordination and superordination.

1. Introduction

Geyer [1] showed that the free surface evolution in a class of third order partial differential equations of moderate amplitude waves in shallow water, all symmetric waves are traveling waves. Diverse investigators utilized various procedures to get the exact traveling wave, some of these techniques are : Jacobi elliptic function expansion process, the homogeneous balance scheme, the sine-cosine technique, adapted simple system of equations, supplementary equation method, generalized shortest algebraic process, the Kudryashov technique, special function formula, rational growth technique, homotopy perturbation procedure, variational method, numerical methods, fractional calculus representations, generalized mapping design and recently by using the conformable calculus (special type of the concept of fractional calculus) [2–5]. Numerical, variational and transformation methods for solving different classes of fractional differential equations can be find in [6–20].

Tasbozan et al. [3] offered a new class of analytic solutions of nonlinear conformable time-fractional coupled Drinfeld–Sokolov–Wilson equation which appears in shallow water flow simulations, when special shapes are utilized to arrange the shallow water equations by means of Sine-Gordon growth technique. The notions of fractional calculus (there are many types of this calculus) have been extensively utilized to formulate real world problems by many researchers in current years. These notions have ability to describe the historical behavior of solutions for all types of differential equations. Therefore, they are indicated as good tools to introduce analytic solutions, attractive solutions, extreme solutions and continuation solutions in complex domains [21].

The complex Ginzburg–Landau equation (see [22–24] for recent work) is the best famous nonlinear equation in physical sciences. It models basically the dynamics of oscillating, spatially generalized systems close to the onset of oscillations. It formulates a huge variety of phenomena from

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nonlinear waves to second-order phase evolution in field theory. Our aim is to introduce a time-space fractional complex Ginzburg–Landau equation using a symmetric conformable concept in both time and space. Our method is based on geometric function theory to establish analytic solutions and get a vision into spatially generalized systems.

By using the modified Riemann–Liouville fractional differential operator

$$D^{\alpha}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}(\phi(\tau)-\phi(0))d\tau,\quad 0\le\alpha<1,$$

the 1D fractional complex Ginzburg–Landau equation is formulated by

$$i\frac{\partial^{\alpha}w}{\partial\tau^{\alpha}}+A\frac{\partial^{2\alpha}w}{\partial\chi^{2\alpha}}=i\,\Phi(w), \tag{1.1}$$

where $0\le\alpha<1$ represents to the fractional order differentiation, χ symbolizes the spatial variable distance along the fiber line while τ is the time in its dimensionless form and $w(t,\chi)$ specifies a complex wave amplitude. The functional Φ acts as the nonlinear formula of the water wave under consideration. Moreover, the 2D fractional complex Ginzburg–Landau (2D-FCGL) equation is formulated by

$$i\frac{\partial^{\alpha}w}{\partial\tau^{\alpha}}+A\frac{\partial^{2\alpha}w}{\partial\chi^{2\alpha}}+B\frac{\partial^{2\alpha}w}{\partial y^{2\alpha}}=i\,\Phi(w), \tag{1.2}$$

where $w=w(t,\chi,y)$. The ordinary cases are discussed in [25,26]. Osman [22] considered the function Φ as the nonlinear evolution equation involving all the partial derivatives of w as follows:

$$i\frac{\partial w}{\partial\tau}+A\frac{\partial^2 w}{\partial\chi^2}+B\frac{\partial^2 w}{\partial y^2}=i\,\Phi(w,w_{\chi},w_{\chi\chi},w_y,w_{yy},w_t,..), \tag{1.3}$$

where $A=B=1/2$.

Recently, connected to the fractional calculus field, Khalil et al. [27] formulated a "conformable fractional derivative" definition of a given real valued function $\phi:[0,\infty)\to\mathbb{R}$ as

$$\mathcal{D}^{\nu}[\phi(t)]=lim_{\epsilon\to0}\frac{\phi(t+\epsilon t^{1-\nu})-\phi(t)}{\epsilon} \tag{1.4}$$

for all $t>0$ and a fractional power $\nu\in(0,1)$. If ϕ is ν-differentiable in some $(0,a), a>0$, and $lim_{t\to0^+}\phi^{\nu}(t)$ exists, then define $\phi^{\nu}(0)=lim_{x\to0^+}\phi^{\nu}(t)$.

A more generalization criteria for differential operators to be a real valued conformable fractional derivative was recently proposed by Anderson and Ulness ([28], Definition 1.1).

Definition 1.1. Conformable Differential Operator. Let ν be a fractional power such that $\nu\in[0,1]$. A differential operator $\mathcal{D}^{\nu}$ is conformable if and only if $\mathcal{D}^0$ is the identity operator and $\mathcal{D}^1$ is the classical differential operator. Specifically, $\mathcal{D}^{\nu}$ is conformable if and only if for differentable function $\phi(t)$,

$$\mathcal{D}^0\phi(t)=\phi(t)\quad and\quad \mathcal{D}^1\phi(t)=\frac{d}{dt}\phi(t)=\phi'(t).$$

In [29], the authors noted that in control theory, a proportional-derivative controller for controlling output μ at time t with two tuning parameters has the algorithm

$$\mu(t)=\kappa_p\Xi(t)+\kappa_d\frac{d}{dt}\Xi(t),$$

where κ_p is the proportional gain, κ_d is the derivative gain, and Ξ is the error between the state variable and the process variable.

A symmetric relation is a kind of binary relation. An instance is the relation "is equal to", because if $x=y$ is true then $y=x$ is likewise correct. Officially, a binary relation R over a set χ is symmetric if and only if: $\forall x,y\in\chi\,(xRy\Leftrightarrow yRx)$. If RT symbolizes the converse of R, then R is symmetric if and only if $R=RT$. In this paper, we improve the idea of conformable calculus to include symmetric property of the function by suggesting the derivative operator

Definition 1.2. Let $\nu\in[0,1]$. A differential operator Δ^{ν} is called a symmetric conformable differential operator if and only if for ϕ is a differential function we have

$$\Delta^{\nu}\phi(t)=\left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\phi'(t)-\left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\phi'(-t). \tag{1.5}$$

such that $\kappa_1(\nu,t)\neq-\kappa_0(\nu,t)$,

$$\lim_{\nu\to0}\kappa_1(\nu,t)=1,\quad \lim_{\nu\to1}\kappa_1(\nu,t)=0,$$
$$\kappa_1(\nu,t)\neq0,\ \forall t,\ \nu\in(0,1),$$

and

$$\lim_{\nu\to0}\kappa_0(\nu,t)=0,\quad \lim_{\nu\to1}\kappa_0(\nu,t)=1,$$
$$\kappa_0(\nu,t)\neq0,\ \forall t,\nu\in(0,1).$$

We have the following properties

Proposition 1.3. *Let $0<\nu<1$ and the symmetric conformable differential operator Δ^{ν} be given as in (2.1). Suppose that the functions ϕ and ψ are differentiable. Then*

1. $\Delta^{\nu}(a\phi+b\psi)=a\Delta^{\nu}\phi+b\Delta^{\nu}\psi$; for all $a,b\in\mathbb{R}$;
2. $\Delta^{\nu}(\kappa)=0$, for all $\kappa\in\mathbb{R}$;
3. $\Delta^{\nu}(\phi\psi)=\phi\Delta^{\nu}(\psi)+\psi\Delta^{\nu}(\phi)$;
4. $\Delta^{\nu}(\phi/\psi)=\frac{\psi\Delta^{\nu}(\phi)-\phi\Delta^{\nu}(\psi)}{\psi^2}$; where $\psi\neq0$.

Proof. For the first item, we have

$$\begin{aligned}\Delta^{\nu}(a\phi+b\psi)(t)&=\left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)(a\phi+b\psi)'(t)\\&\quad-\left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)(a\phi+b\psi)'(-t)\\&=a\Bigg[\left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\phi'(t)\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\phi'(-t)\Bigg]\\
&+b\Bigg[\left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\psi'(t)\\
&-\left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\psi'(-t)\Bigg]\\
&=a\Delta^\nu\phi(t)+b\Delta^\nu\psi(t).
\end{aligned}
$$

The second item is obtained directly. The third item can prove as follows:

$$
\begin{aligned}
&\Delta^\nu(\phi\psi)(t)\\
&=\left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)(\phi\psi)'(t)\\
&-\left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)(\phi\psi)'(-t)\\
&=\left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)(\phi\psi'+\psi\phi')(t)\\
&-\left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)(\phi\psi'+\psi\phi')(-t)\\
&=\phi\Bigg[\left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\psi'(t)\\
&-\left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)(\psi')(-t)\Bigg]\\
&+\psi\Bigg[\left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\phi'(t)\\
&-\left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\phi')(-t)\Bigg]\\
&=\phi\Delta^\nu(\psi)+\psi\Delta^\nu(\phi).
\end{aligned}
$$

Finally, for the division, we have

$$
\begin{aligned}
&\Delta^\nu(\phi/\psi)(t)\\
&=\left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)(\phi/\psi)'(t)-\left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)(\phi/\psi)'(-t)\\
&=\left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\left(\frac{\phi'\psi-\phi\psi'}{\psi^2}\right)(t)\\
&-\left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\left(\frac{\phi'\psi-\phi\psi'}{\psi^2}\right)(-t)\\
&=\frac{\psi\Big[\left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\phi'(t)-\left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\phi')(-t)\Big]}{\psi^2}\\
&-\frac{\phi\Big[\left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)\psi'(t)-\left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right)(\psi')(-t)\Big]}{\psi^2}\\
&=\frac{\psi\Delta^\nu(\phi)-\phi\Delta^\nu(\psi)}{\psi^2}.
\end{aligned}
$$

□

2. Complex conformable differential operator

The following basic information are represented to concepts that will utilize throughout this paper. A function $w\in\Lambda$ is called univalent in $\mathbb{U}$ if it never acts the same value twice; that is, if $\xi_1\neq\xi_2$ in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ then $w(\xi_1)\neq w(\xi_2)$ or equivalently, if $w(\xi_1)=w(z_2)$ then $\xi_1=\xi_2$. Without loss of generality, we may indicate the notion Λ for our univalent functions with the following series

$$w(z)=z+\sum_{n=2}^{\infty}w_n z^n,\quad z\in\mathbb{U}. \tag{2.1}$$

We let $\mathcal{S}$ denote the class of such functions $w\in\Lambda$ that are univalent in $\mathbb{U}$ (e.g. see Duren [30]). The Bieberbach conjecture tells us that $|w_n|\leq n$, $\forall n\geq 2$. Lowner [31] introduced a new technique for univalent functions by suggesting the concept of parametric univalent function via the following given differential equation, which is known as a Loewner–Kufarev control system

$$\frac{d}{dt}w_t(z)=\Phi(w_t(z)),\quad w_0(z)=z\in\mathbb{U},\ t\geq 0,$$

where Φ is a holomorphic function called the (infinitesimal) generator of w. In general, the functional $w_t(z)$ takes the following formula

$$
\begin{aligned}
w_t(z)=\frac{z}{(1-tz)^2}&=z+2tz^2+3t^2z^3+4t^3z^4+5t^4z^5\\
&+6t^5z^6+O(z^7),
\end{aligned}\tag{2.2}
$$

$$\left(t<\frac{1}{|z|},z\neq 0,\ z\in\mathbb{U}\right).$$

Eq. (2.3) is called the parametric Koebe function. The Koebe function maps the unit disk $|z|<1$ onto the complex plane with a slit along the ray beginning at the point with radius 1/4, its extension involving the point $z=0$. The Koebe function is an extreme function of the class of convex univalent functions. We shall suggest a solution of our water wave equation by using the rotate Koebe function of the form (see Fig. 2.1)

$$
\begin{aligned}
w_t(z)=\frac{z}{(1-e^{it}z)^2}&=z+2e^{it}z^2+3e^{2it}z^3+4e^{3it}z^4\\
&+5e^{4it}z^5+6e^{5it}z^6+O(z^7),\quad z\in\mathbb{U}.
\end{aligned}\tag{2.3}
$$

It is well known that

$$\frac{r}{(1+r)^2}\leq|w_t(z)|\leq\frac{r}{(1-r)^2}$$

and

$$\frac{1-r}{(1+r)^3}\leq|w_t'(z)|\leq\frac{1+r}{(1-r)^3}.$$

Berkson and Porta [32] suggested the functional, for some analytic functions $\wp$ in $\mathbb{U}$

$$\Phi(\wp(z))=(\xi-z)(1-\bar{\xi}z)\wp(z),\ z\in\mathbb{U},\ \xi\in\bar{\mathbb{U}}.$$

The univalent solution is very important in wave equations (see [33]). It is well known that, the solutions of wave equations certainly are invalid for an infinite layer because they will not be univalent functions (the peaks of the wave will reliably travel faster than the through and lastly reach these levels). Our problem will subject to the boundary condition $w\in\mathcal{S}\subset\Lambda$. In this case, we able to study the connection problem (coefficient estimate) of the solution of the wave

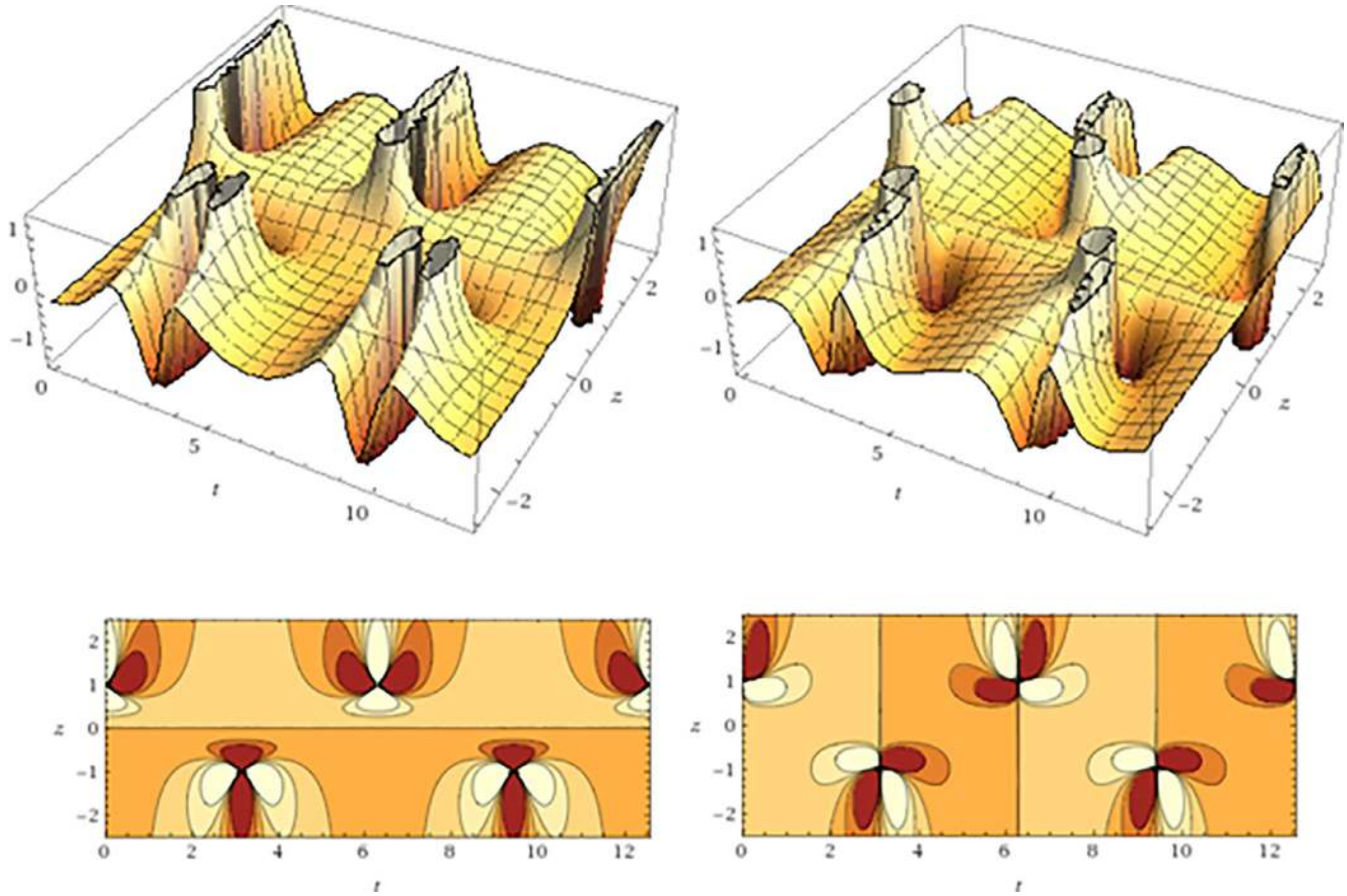

Fig. 2.1. Real and imaginary 3D- $w_t(z) = \frac{z}{(1-e^{it}z)^2}$.

equation by discussing the corresponding connection problem of $w \in \mathcal{S}$. We extend the definition (1.5) to the z-plane.

Definition 2.1. For a function $w(z) \in \Lambda$, and a constant $\nu \in [0, 1]$, we formulate the symmetric conformable differential operator as follows:

$$\begin{aligned}
\Delta^{0\nu} w(z) &= w(z) \\
\Delta^{\nu} w(z) &= \left(\frac{\kappa_1(\nu, z)}{\kappa_1(\nu, z) + \kappa_0(\nu, z)}\right) z w'(z) \\
&\quad - \left(\frac{\kappa_0(\nu, z)}{\kappa_1(\nu, z) + \kappa_0(\nu, z)}\right) z w'(-z) \\
&= \left(\frac{\kappa_1(\nu, z)}{\kappa_1(\nu, z) + \kappa_0(\nu, z)}\right)\left(z + \sum_{n=2}^{\infty} n w_n z^n\right) \\
&\quad - \left(\frac{\kappa_0(\nu, z)}{\kappa_1(\nu, z) + \kappa_0(\nu, z)}\right)\left(-z + \sum_{n=2}^{\infty} n(-1)^n w_n z^n\right) \\
&= z + \sum_{n=2}^{\infty} n\left(\frac{\kappa_1(\nu, z) + (-1)^{n+1}\kappa_0(\nu, z)}{\kappa_1(\nu, z) + \kappa_0(\nu, z)}\right) w_n z^n \\
&:= z + \sum_{n=2}^{\infty} W_n z^n,
\end{aligned}$$

$$W_n := n\left(\frac{\kappa_1(\nu, z) + (-1)^{n+1}\kappa_0(\nu, z)}{\kappa_1(\nu, z) + \kappa_0(\nu, z)}\right) w_n \tag{2.4}$$

for the second fractional derivative, we calculate it as follows:

$$\begin{aligned}
\Delta^{2\nu} w(z) &= \Delta^{\nu}[\Delta^{\nu} w(z)] = \Delta^{\nu}[z + \sum_{n=2}^{\infty} W_n z^n] \\
&= \left(\frac{\kappa_1(\nu, z)}{\kappa_1(\nu, z) + \kappa_0(\nu, z)}\right)\left(z + \sum_{n=2}^{\infty} n W_n z^n\right) \\
&\quad - \left(\frac{\kappa_0(\nu, z)}{\kappa_1(\nu, z) + \kappa_0(\nu, z)}\right)\left(-z + \sum_{n=2}^{\infty} n(-1)^n W_n z^n\right) \\
&= z + \sum_{n=2}^{\infty} n\left(\frac{\kappa_1(\nu, z) + (-1)^{n+1}\kappa_0(\nu, z)}{\kappa_1(\nu, z) + \kappa_0(\nu, z)}\right) W_n z^n \\
&= z + \sum_{n=2}^{\infty} n^2\left(\frac{\kappa_1(\nu, z) + (-1)^{n+1}\kappa_0(\nu, z)}{\kappa_1(\nu, z) + \kappa_0(\nu, z)}\right)^2 w_n z^n
\end{aligned}$$

In general, we compute the kth fractional derivative as follows:

$$\begin{aligned}
\Delta^{k\nu} w(z) &= \Delta^{\nu}[\Delta^{(k-1)\nu} w(z)] \\
&= z + \sum_{n=2}^{\infty} n^k\left(\frac{\kappa_1(\nu, z) + (-1)^{n+1}\kappa_0(\nu, z)}{\kappa_1(\nu, z) + \kappa_0(\nu, z)}\right)^k w_n z^n.
\end{aligned}$$

so that $\kappa_1(\nu, z) \neq -\kappa_0(\nu, z)$,

$$\lim_{\nu \to 0} \kappa_1(\nu, z) = 1, \quad \lim_{\nu \to 1} \kappa_1(\nu, z) = 0, \quad \kappa_1(\nu, z) \neq 0,$$

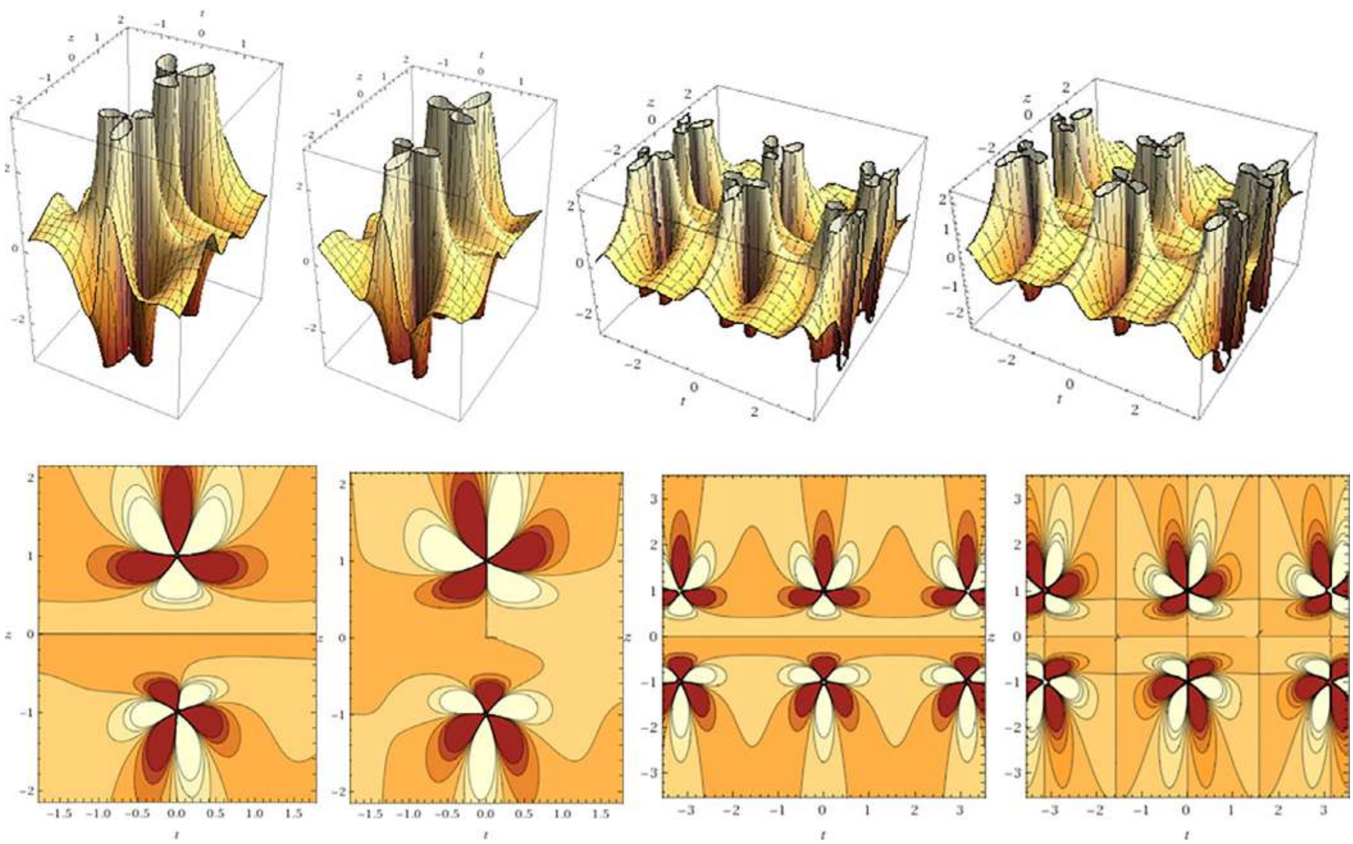

Fig. 2.2. Real and imaginary 3D- $\Delta^{0.25}\left(\frac{z}{(1-e^{it}z)^2}\right)_z$ and $\Delta^{0.5}\left(\frac{z}{(1-e^{it}z)^2}\right)_z$ respectively.

$$\forall z \in \mathbb{U}, \ \nu \in (0,1),$$

and

$$\lim_{\nu\to 0}\kappa_0(\nu,z)=0, \quad \lim_{\nu\to 1}\kappa_0(\nu,z)=1, \quad \kappa_0(\nu,z)\neq 0, \quad \forall z\in\mathbb{U}\ \nu\in(0,1).$$

It is clear that when $k=0$, we have

$$\Delta^0 w(z) = z+\sum_{n=2}^{\infty} w_n z^n = w(z), \quad z\in\mathbb{U}.$$

Moreover, in terms of the convolution product (*) of analytic functions, $\Delta^{k\nu}w(z), z\in\mathbb{U}$ reduces to the formula

$$\Delta^{\nu}w(z) = \left(z+\sum_{n=2}^{\infty} n^k\left(\frac{\kappa_1(\nu,z)+(-1)^{n+1}\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\right)^k z^n\right) * \left(z+\sum_{n=2}^{\infty} w_n z^n\right).$$

For example (see Fig. 2.2),

$$\begin{aligned}\Delta^{0.25}\left(\frac{z}{(1-e^{it}z)^2}\right)_z &= z+4e^{(it)}z^2-8/3e^{(it)}z^{(5/2)}+((8e^{(it)})/9+9e^{(2it)})z^3\\ &-8/27e^{(it)}z^{(7/2)}+((8e^{(it)})/81+16e^{(3it)})z^4\\ &-8/243(e^{(it)}(1+324e^{(2it)}))z^{(9/2)}+((8e^{(it)})/729\\ &+32/9e^{(3it)}+25e^{(4it)})z^5\\ &-(8(e^{(it)}(1+324e^{(2it)}))z^{(11/2)})/2187+O(z^6) \end{aligned} \tag{2.5}$$

and with respect to t, we obtain

$$\begin{aligned}\Delta^{0.25}\left(\frac{z}{(1-e^{it}z)^2}\right)_t &= (2ie^{(-it)}(\sqrt{(t)}+3e^{(2it)})z^2)/(\sqrt{(t)}+3)\\ &+(6ie^{(-2it)}(\sqrt{(t)}+3e^{(4it)})z^3)/(\sqrt{(t)}+3)\\ &+(12ie^{(-3it)}(\sqrt{(t)}+3e^{(6it)})z^4)/(\sqrt{(t)}+3)\\ &+(20ie^{(-4it)}(\sqrt{(t)}+3e^{(8it)})z^5)/(\sqrt{(t)}+3)\\ &+(30ie^{(-5it)}(\sqrt{(t)}+3e^{(10it)})z^6)/(\sqrt{(t)}+3)+O(z^7) \end{aligned} \tag{2.6}$$

In the similar manner of Proposition 1.3, we have the following result:

Proposition 2.2. *Let $0<\nu<1$ and the complex symmetric conformable differential operator Δ^{ν} be given as in (2.4). Consider two functions $w, u \in \Lambda$. Then*

1. $\Delta^{\nu}(aw+bu)=a\Delta^{\nu}w+b\Delta^{\nu}u$; for all $a,b\in\mathbb{C}$;
2. $\Delta^{\nu}(\kappa)=0$, for all $\kappa\in\mathbb{C}$;
3. $\Delta^{\nu}(wu)=w\Delta^{\nu}(u)+u\Delta^{\nu}(w)$;
4. $\Delta^{\nu}(w/u)=\frac{u\Delta^{\nu}(w)-w\Delta^{\nu}(u)}{u^2}$; where $u\neq 0$.

We proceed to formulate the following problem: Let $0<\nu<1$ be the fractional order power, z be a 2D domain in the open unit disk $\mathbb{U}$, while t be the time in its dimensions form and $w_t(z)$ be a complex wave amplitude. The functional Φ acts as the nonlinear formula of the water wave under consideration. Moreover, the 2D fractional complex conformable Ginzburg–Landau equation is formulated by

$$i\,\Delta_t^{\nu} w_t(z) + A\Delta_z^{2\nu} w_t(z) = i\Phi(w_t(z)), \quad A \in \mathbb{C} \tag{2.7}$$

and by letting $A = i$, we have the 2D-FCCGL

$$\Delta_t^{\nu} w_t(z) + \Delta_z^{2\nu} w_t(z) = \Phi(w_t(z)), \tag{2.8}$$

where

$$\Delta_t^{\nu} w_t(z) = \left(\frac{\kappa_1(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right) w_t'(z) - \left(\frac{\kappa_0(\nu,t)}{\kappa_1(\nu,t)+\kappa_0(\nu,t)}\right) w_{(-t)}'(z)$$

and

$$\Delta_z^{2\nu} w_t(z) = \Delta^{\nu}[\Delta^{\nu} w_t(z)] = \Delta^{\nu}\left[z + \sum_{n=2}^{\infty} W_n z^n\right] = z + \sum_{n=2}^{\infty} n^2 \left(\frac{\kappa_1(\nu,z)+(-1)^{n+1}\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\right)^2 w_n z^n.$$

Moreover, we consider a fractional functional $\Phi(w_t(z))$ based on a bilinear transformation in the open unit disk. The conformable functions κ_0 and $_i$ are given in the following shape

$$\kappa_0(\nu,t) = \nu t^{1-\nu}, \quad \kappa_0(\nu,z) = \nu z^{1-\nu}$$

and

$$\kappa_1(\nu,t) = (1-\nu)t^{\nu}, \quad \kappa_0(\nu,z) = (1-\nu)z^{\nu}$$

for all $z \in \mathbb{U}$ and $t \in (0, \infty)$. In this place, we note that Eq. (2.8) is a modification style of Eq. (1.2), by combining the 2-D axis in z-plane. Therefore, we select $A = B = i$.

Our methodology is based on the idea of majorization of connection problems (coefficients estimate). For $h(z) = \sum \ell_n z^n$ and $\hbar(z) = \sum \flat_n z^n$, $\flat_n \geq 0$ for all $n \geq 0$ we have $h \ll \hbar$ if and only if $|\ell_n| \leq \flat_n$. In this place, we note that there is a sound connection between majorization and subordination ($f \prec g$) in the geometric function theory. It has been shown that, under some conditions the subordination becomes the majorization. Furthermore, the majorization is represented to the upper bound of solutions of differential equations. We shall use this method to approximate the solution of Eq. (2.8) to well known functions. The advantageous of the approximation are firstly, for distinguishing objective functions, where the approximation method studies how fixed known functions can be approximated by a certain class of functions that often have required properties. Secondly, the objective function, say it Σ, may be unidentified; instead of a usual express, only a set of points of the form $(x, \Sigma(x))$ is supplied. Subject to the association of the domain and codomain of Σ, numerous approaches for approximating g may be appropriate. For example, if Σ is any type of operations (differential, integral or linear operators) on the z - plane, methods of geometric function theory can be utilized (approximation by using convex classes of analytic functions [34], subordination and superordination methods [35], or majorization by using connection problem [36]).

3. Analytic solutions

In this section, we proceed to construct the analytic solutions of the wave Eq. (2.8). An analytic solution $w_t(z)$ of (2.8) is called attractive analytic solution if and only if the term $\Delta_\tau^{\nu} w_t(z) + \Delta_z^{2\nu} w_t(z)$ is majorized by the functional $\Phi(w_t(z))$, that is $|c_n| \leq \varphi_n$, where $\Delta_\tau^{\nu} w_t(z) + \Delta_z^{2\nu} w_t(z) = \sum c_n z^n$ and $\Phi(w) = \sum \varphi_n z^n, \varphi > 0, \forall n$. Since the left hand of Eq. (2.8) involves a fractional power in terms of $0 < \nu < 1$ therefore, the Berkson–Porta functional $\Phi(w_t(z)) = (\xi - z)(1 - \bar{\xi} z) w_t(z)$, $z \in \mathbb{U}$ does not satisfy the majority condition because of vanishing the fractional power terms. Thus, we suggest a fractional functional $\Phi(w_t(z))$ in terms of the general bilinear transformation

$$J^{\beta}(z) = \left(\frac{1+\zeta z}{1-\xi z}\right)^{\beta}, \quad z \in \mathbb{U},\ \zeta, \xi \in \bar{\mathbb{U}},\ \beta \in [0, \infty).$$

Based on the definition of $J^{\beta}(z)$, we formulate the functional $\Phi(w_t(z))$ as follows:

$$\Phi^{\nu}(w_t(z)) = \left(\frac{1+\sqrt{z}}{1-z}\right)^{\nu} w_t(z), \tag{3.1}$$

$$\left(z \in \mathbb{U},\ 0 < \nu < 1, \zeta = \frac{1}{\sqrt{z}}, \xi = 1\right),$$

having the expansions, with the help of Wolfram Alpha–Mathematica 12.00 (see Fig. 3.1)

$$\begin{aligned}\left(\frac{1+\sqrt{z}}{1-z}\right)^{0.25} w_t(z) &= z + 0.25z^{3/2} + z^2(0.15625 + 2e^{it}) \\ &+ z^{5/2}(0.117188 + 0.5e^{(it)}) + z^3(0.0952148 \\ &+ 0.3125e^{(it)} + 3e^{(2it)}) + z^{(7/2)}(0.0809326 \\ &+ 0.234375e^{(it)} + 0.75e^{(2it)}) \\ &+ z^4(0.070816 + 0.19043e^{(it)} + 0.46875e^{(2it)} + 4e^{(3it)}) \\ &+ z^{(9/2)}(0.0632286 + 0.161865e^{(it)} + 0.351563e^{(2it)} + e^{(3it)}) \\ &+ z^5(0.0573009 + 0.141632e^{(it)} + 0.285645e^{(2it)} \\ &+ 0.625e^{(3it)} + 5e^{(4it)}) + z^{(11/2)}(0.0525258 + 0.126457e^{(it)} \\ &+ 0.242798e^{(2it)} + 0.46875e^{(3it)} + 1.25e^{(4it)}) + O(z^6)\end{aligned} \tag{3.2}$$

Eq. (3.2) can be approximated by taking the total sum of roots with respect to the parametric coefficients in terms of t as follows

$$\begin{aligned}\Phi^{0.25}(w_t(z)) \approx\ & z + 0.25z^{(3/2)} + 9.5z^2 + 9.5z^{(5/2)} + 12.5z^3 \\ &+ 12.5z^{(7/2)} + 12.5z^4 + 18.5z^{(9/2)} + 19.0z^5 \\ &+ 19.0z^{(11/2)} + O(z^6)\end{aligned} \tag{3.3}$$

Similarly, we have

$$\begin{aligned}\left(\frac{1+\sqrt{z}}{1-z}\right)^{0.5} w_t(z) &= z + z^{(3/2)}/2 + (3/8 + 2e^{(it)})z^2 \\ &+ (5/16 + e^{(it)})z^{(5/2)} + ((3e^{(it)})/4 + 3e^{(2it)} + 35/128)z^3\end{aligned}$$

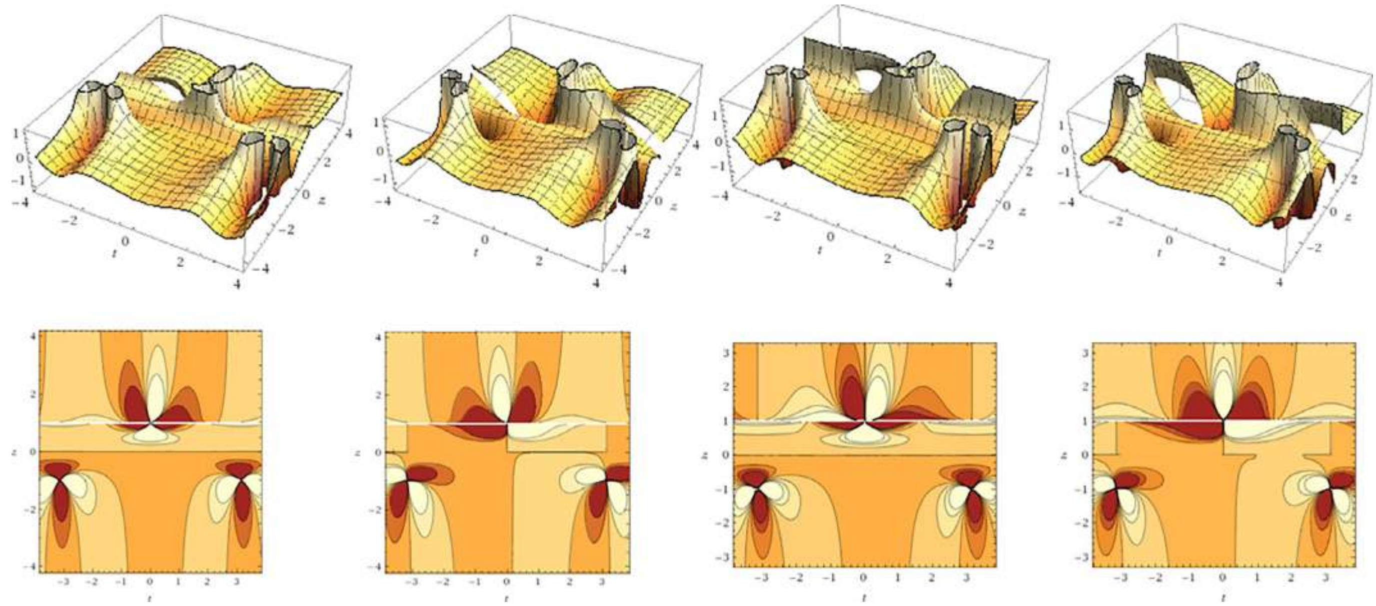

Fig. 3.1. Real and imaginary 3D plot of $\Phi^{\nu}(w_t(z))$ for $\nu = 0.25$ and $\nu = 0.5$, respectively.

$$
\begin{aligned}
&+((5e^{(it)})/8 + 3/2e^{(2it)} + 63/256)z^{(7/2)} + ((35e^{(it)})/64\\
&+9/8e^{(2it)} + 4e^(3it) + 231/1024)z^4 + ((63e^{(it)})/128\\
&+15/16e^{(2it)} + 2e^{(3it)} + 429/2048)z^{(9/2)} + ((231e^{(it)})/512\\
&+105/128e^{(2it)} + 3/2e^{(3it)} + 5e^{(4it)} + 6435/32768)z^5\\
&+((429e^{(it)})/1024 + 189/256e^{(2it)} + 5/4e^{(3it)} + 5/2e^{(4it)}\\
&+12155/65536)z^{(11/2)} + O(z^6) \qquad (3.4)
\end{aligned}
$$

and its approximation

$$
\begin{aligned}
\Phi^{0.5}(w_t(z)) \approx\ & z + z^{(3/2)}/2 + 9.5z^2 + 9.5z^{(5/2)} + 9.5z^3\\
&+12.5z^{(7/2)} + 17.5z^4 + 17.5z^{(9/2)}\\
&+24z^5 + 24z^{11/2} + O(z^6) \qquad (3.5)
\end{aligned}
$$

Proposition 3.1. *Consider the wave Eq. (2.8) with the fractional functional $\Phi^{\nu}(w_t(z))$, $\nu \in (0, 1)$ given in (3.1). Then the rotated Koebe function $w_t(z) = \frac{z}{(1-e^{it}z)^2}$ is an attractive solution for (2.8).*

Proof. By the symmetric behavior of Δ^{ν}, we study two cases when $\nu = 0.25$ and $\nu = 0.5$. A computation implies that

$$
\begin{aligned}
&\Delta_t^{0.25}\frac{z}{(1-e^{it}z)^2} + \Delta_z^{2(0.25)}\frac{z}{(1-e^{it}z)^2}\\
&= \Delta_t^{0.25}\frac{z}{(1-e^{it}z)^2} + \Delta_z^{0.25}(\Delta_z^{0.25}\frac{z}{(1-e^{it}z)^2})\\
&= 3/(3+\sqrt{(t)})d/(dt)(z/(1-e^{(it)}z)^2)\\
&\quad -\sqrt{(t)}/(3+\sqrt{(t)})d/(dt)z/(1-e^{(-it)}z)^2\\
&\quad +\Delta_z^{0.25}((3/(3+z^{0.5}))*(z)*(d/dz)(z/(1-e^{(it)}z)^2)\\
&\quad -(z^{0.5}/(3+z^{0.5}))(z)(d/dz)(-z/(1+e^{(it)}z)^2))\\
&\approx z + 2.5z^2 + 2z^3 + 2z^4 + 5.5z^{(9/2)} + 2z^5 + 5.5z^{(11/2)} + O(z^6) \qquad (3.6)
\end{aligned}
$$

Comparing Eqs. (3.3) and (3.6), we conclude that $\Delta_t^{0.25}\frac{z}{(1-e^{it}z)^2} + \Delta_z^{2(0.25)}\frac{z}{(1-e^{it}z)^2}$ is majorized by the function $\Phi^{0.25}(w_t(z))$. Similarly, when $\nu = 0.5$, we find that

$$
\begin{aligned}
&\Delta_t^{0.5}\frac{z}{(1-e^{it}z)^2} + \Delta_z^{2(0.5)}\frac{z}{(1-e^{it}z)^2}\\
&= \Delta_t^{0.5}\frac{z}{(1-e^{it}z)^2} + \Delta_z^{0.5}\left(\Delta_z^{0.5}\frac{z}{(1-e^{it}z)^2}\right)\\
&= z + z^{(3/2)}/2 + (3/8 + 2e^{(it)})z^2 + (5/16 + e^{(it)})z^{(5/2)}\\
&\quad +((3e^{(it)})/4 + 3e^{(2it)} + 35/128)z^3 + ((5e^{(it)})/8\\
&\quad +3/2e^{(2it)} + 63/256)z^{(7/2)} + ((35e^{(it)})/64 + 9/8e^{(2it)}\\
&\quad +4e^{(3it)} + 231/1024)z^4 + ((63e^{(it)})/128 + 15/16e^{(2it)}\\
&\quad +2e^{(3it)} + 429/2048)z^{(9/2)} + ((231e^{(it)})/512\\
&\quad +105/128e^{(2it)} + 3/2e^{(3it)} + 5e^{(4it)} + 6435/32768)z^5\\
&\quad +((429e^{(it)})/1024 + 189/256e^{(2it)} + 5/4e^{(3it)} + 5/2e^{(4it)}\\
&\quad +12155/65536)z^{(11/2)} + O(z^6)\\
&\approx z + z^{(3/2)}/2 + 9.5z^2 + 9.5z^{(5/2)} + 9.5z^3 + 12.7z^{(7/2)}\\
&\quad +17.6z^4 + 17.5z^{(9/2)} + 24z^5 + O(z^6) \qquad (3.7)
\end{aligned}
$$

Thus, $\Delta_t^{0.5}\frac{z}{(1-e^{it}z)^2} + \Delta_z^{2(0.5)}\frac{z}{(1-e^{it}z)^2}$ is majorized by the function $\Phi^{0.5}(w_t(z))$. And it is true for all $0 < \nu < 1$. □

Proposition 3.2. *Consider the wave Eq. (2.8) with the fractional functional $\Phi^{\nu}(w_t(z))$, $\nu \in (0, 1]$ given in (3.1). Then there is a probability measure μ on $(\partial\mathbb{U})^2$, for $\nu \to 1$ such that $\int_{(\partial\mathbb{U})^2}\Phi(z)d\mu$ exists.*

Proof. Let $\rho, \varrho \in \partial\mathbb{U}$ such that $\rho = 1/\sqrt{z}, |z| < 1$ then $|\rho| = 1$ and

$$
\begin{aligned}
\left(\frac{1+\rho z}{1+\varrho z}\right)^{\nu} &= \frac{(1+z^{0.5})^{\nu}}{1+\varrho z}.\frac{1}{(1+\varrho z)^{\nu-1}}, \nu \to 1\\
&\ll \frac{(1+z^{0.5})^{\nu}}{1-z}.\frac{1}{(1-z)^{\nu-1}} \qquad (3.8)\\
&= \left(\frac{1+z^{0.5}}{1-z}\right)^{\nu}, \quad \nu \to 1.
\end{aligned}
$$

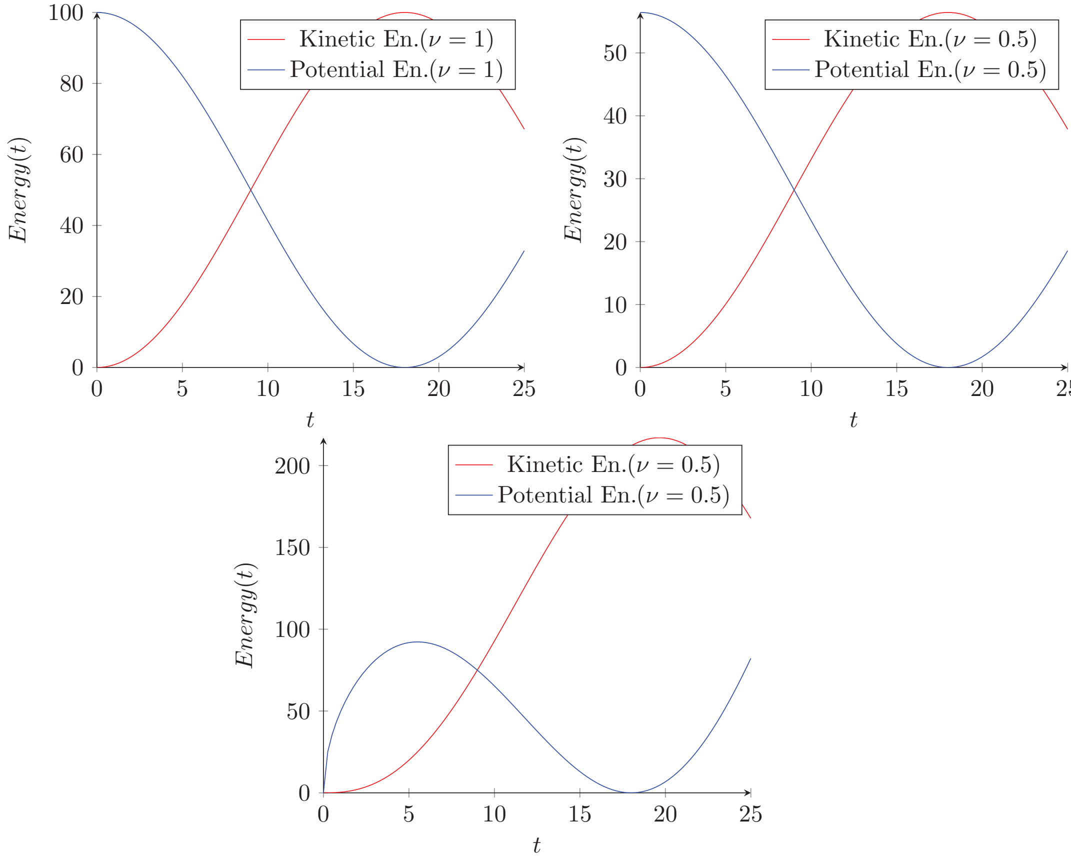

Fig. 3.2. From the left: the behavior of solutions for the original wave equation when $\nu = 1$, followed by the solution of Eq. (1.2) with $\nu = 0.5$ and the last graph is about the solution of Eq. (2.8) for $\nu = 0.5$

Table 3.1
A comparison of solutions with respect to Kinetic & Potential energy (the unit is in Joules) (for $\nu = 0.5, t \in (0, 1)$) and mass=100 g

Wave Eq.	Kinetic En.	Potential En.	Total En.(J)
Original ($\nu = 1$)	0.09	0.008	0.1
2D- FCGL(see Eq. (1.2))	0.029	0.071	0.1
2D- FCCGL(see Eq. (2.8))	0.05	0.05	0.1

In view of Theorem 1.11 in [37], the $\left(\frac{1+\rho z}{1+\varrho z}\right)^{\nu}$ admits a probability measure μ in $(\partial\mathbb{U})^2$ achieving

$$\phi(z) = \int_{(\partial\mathbb{U})^2} \left(\frac{1+\rho z}{1+\varrho z}\right)^{\nu} d\mu(\rho, \varrho), \quad z \in \mathbb{U}.$$

Then according to Proposition 3.1, there is a constant κ (diffusion constant) satisfying

$$\int_{(\partial\mathbb{U})^2} \left(\frac{1+\rho z}{1+\varrho z}\right)^{\nu} d\mu(\rho, \varrho)$$
$$= \kappa \int_{(\partial\mathbb{U})^2} \left(\frac{1+\rho z}{1-\varrho z}\right)^{\nu} w_t(z) d\mu(\rho, \varrho), \quad z \in \mathbb{U}$$

or $\phi(z) = \kappa \int_{(\partial\mathbb{U})^2} \Phi(z) d\mu(\rho, \varrho)$ exists. □

Table 1 together with Fig. 3.2 show the comparison among the solution of original wave equation, 2D- FCGL(see Eq. (1.2)) and 2D- FCCGL(see Eq. (2.8)). Note that the solution of (1.2) is given by the gamma function ($\Gamma(\nu)$) while the solution of (2.8) is given by $\exp(\nu)$. Our comparison is based on finding the energy of the wave for each case. The steps of Algorithm is given in Fig. 3.3.

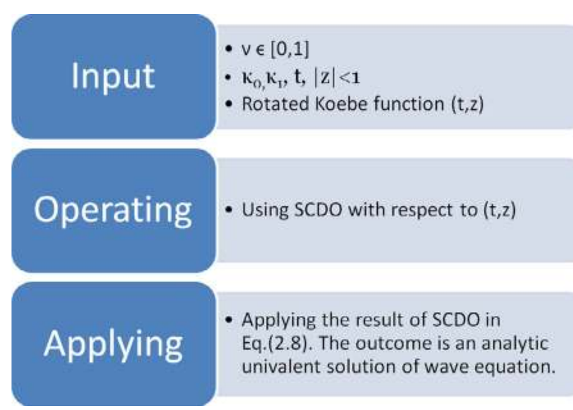

Fig. 3.3. The algorithm of analytic solution of Eq. (2.8).

3.1. Achievements in ocean engineering

The solution of 3D wave equation in (2.8) achieves important sectors of ocean engineering. The newest realizations correspond to improvements of stability, energy conservation, accuracy, boundary conditions and upgraded models of surface tension, multiphase flows, fluid-structure interactions, etc. Notwithstanding the progressions of this equation has made, numerous important features quiet continue to be not systematically determined. Examples of such remaining unresolved problems, omit stability, convergence, adaptability, boundary conditions and consistency or conservation. The stability problem is of critical significance for the stable and reliable application of particle systems to engineering problems involving those come across in ocean engineering. Precisely, the numerical uncertainties related to particle approaches considered into two major groups of rank insufficiency and stress state instabilities. The rank insufficiency variability connected to spurious singular or zero-energy styles happening when the field variables and their derivatives calculated at the same calculation points. In the recent equation, we have equality in both the kinetic and potential energies (see Table 1). Other issues in different ways depend on the conclusion of the energy; for example, the accuracy has a deep connection with the value of the energy (high energy⇔ high accuracy).

Another factor can recognize is that the improvement of boundary conditions. Adam et al. [38] offered a general wall boundary condition, which properly inflicts no-slip conditions even for complex geometries. Notwithstanding being comparatively simple for application, presentation of mirror elements may chief to imprecisions in the convergence of differential operator simulations (see [39]). A more preferred and new approach is connected to growth of so-called semi-analytical boundary conditions. The recent method implies analytic solutions in the open unit disk (the boundary is equal to 1), which means the improvement of the dynamic boundary condition of the wave equation. This type of boundary condition can be classified under the solid boundary condition (conservative of volume, energy and momentum) in wave equations.

3.2. Future works in ocean engineering

The stability of this method is likewise desired to be improved by modification of differential operator schemes, such as Laplacian, gradient, Poisson and other useful operators, agreeing with the expressions that straight, perform in the careful leading equations, or done by applications of higher-order correct numerical solution processes. As for accuracy, in spite of major developments, the issue of un-physical pressure fluctuations relics to be not completely determined. Additional improvements in accuracy estimated to achieve obligations to the deep and careful investigations that are actuality lead in this area.

4. Conclusion

Encouraged by using the analytic representation method, in the current analysis, we have considered a 2D- class of water wave equations in a complex domain. This class is included a symmetric conformable differential operator for time and space variables. We formulated new symmetric conformable operators (real and complex). We discussed some of the basic properties of these operators. The analytic solution is offered by using the memorization concept. The showed that 2D- FCCGL(see Eq. (2.8)) has equaled Kinetic and Potential energy, this is because of the symmetric operator.

In the sequel, we used Mathematica 12.0.0 (2019) to obtain all our numerical computations.

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Declaration of Competing Interest

The authors declare no conflict of interest.

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