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An Avant-Garde Handling of Temporal-Spatial Fractional Physical Models

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Abstract: In the present study, we dilate the differential transform scheme to develop a reliable scheme for studying analytically the mutual impact of temporal and spatial fractional derivatives in Caputo's sense. We also provide a mathematical framework for the transformed equations of some fundamental functional forms in fractal 2-dimensional space. To demonstrate the effectiveness of our proposed scheme, we first provide an elegant scheme to estimate the (mixed-higher) Caputo-fractional derivatives, and then we give an analytical treatment for several (non)linear physical case studies in fractal 2-dimensional space. The study concluded that the proposed scheme is very efficacious and convenient in extracting solutions for wide physical applications endowed with two different memory parameters as well as in approximating fractional derivatives.

Keywords: Caputo fractional derivative, PDEs in fractal 2D space, twofold-fractional differential transform

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1 Introduction

Many physical phenomena may possess indeterminate memory and heritage properties due to the non-locality nature of the power-law structure. Examples of these include the universal electromagnetic and mechanical responses [1], the viscoelastic responses of hydropolymers [2], the flux of the viscous fluid flowing through the porous medium [3], and the diffusion of stress waves in viscoelastic media [50]. Intensive studies have been carried out and found that these memory and heritage characteristics can be retrieved by converting the integer-derivatives into fractional-derivatives [1–3, 5–18, 50]. As a result, mathematical models with fractal derivatives have emerged, and a scientific expedition started with developing and adapting mathematical methods for seeking solutions of these hybrid models. Instances of such existing methods are Galerkin finite element method [19, 20], spectral collocation methods [21–23], homotopy analysis/perturbation/asymptotic methods [24–27], variational iteration method [28, 29], Laplace–Fourier transform methods [30, 31], Adomian decomposition method [32–34], differential transform method [35–37], and modified Taylor power series method [38–49].

Most of the aforementioned methods were intended to study the effects of the memory parameter on either the temporal or the spatial coordinate. Recently, an adaptation of Taylor's series solution method has been provided to examine the joint effect of the temporal and spatial fractional derivatives simultaneously [41]. The method depends predominantly on a new fractional power series representation (FPSR) that involves two fractional derivative parameters. Continuing with this trend, we integrate this FPSR with the traditional differential transform idea to provide a convenient treatment for temporal-spatial fractional models. It should be noted here that our study focuses on examining the initial value problems that equipped with analytic initial conditions in a fractional sense (i. e. the initial conditions have a fractional power series representation). Subsequent work will be devoted to further development of the method to provide an approximate solution of these hybrid models with boundary conditions by adapting the idea presented in [50].

It is notable that there is no unified form for a fractional derivative. However, for the aim of this work, we embrace the Caputo sense, which is known for a suitable function $u(t,x)$ as:

$$\mathcal{D}_t^{\gamma}\left[u(t,x)\right] = \frac{1}{\Gamma(m-\gamma)}\int_0^t (t-\kappa)^{m-\gamma-1}\frac{\partial^m u(\kappa,x)}{\partial\kappa^m}d\kappa, \quad (1)$$

where $m-1<\gamma<m$ is the fractional derivative order. For simplicity, we may assume that $\gamma \in (0,1)$ since for any $m-1<\gamma<m \in \mathbb{N}$, we can write $\mathcal{D}_t^{\gamma}\left[u(t,x)\right] = \mathcal{D}_t^{\gamma-(m-1)}\left[u^{(m-1)}(t,x)\right]$ with $0<\gamma-(m-1)<1$. A straightforward application of formula (1) with $\gamma\in(0,1)$ yields:

$$\mathcal{D}_t^{\gamma}\left[t^a\right] = \begin{cases} \dfrac{\Gamma(a+1)}{\Gamma(a-\gamma+1)}t^{a-\gamma}; & a>0 \\ 0 & ; \; a=0. \end{cases} \quad (2)$$

We refer the reader to [11–14] for additional information about fractional calculus and its applications.

The rest of the paper is organized as follows. In the next section, we extend the idea of the differential transform method to involve twofold Caputo-fractional derivatives. In Section 3, we present two significant usages for the extension scheme. Finally, we conclude with some remarks in Section 4.

2 Twofold-fractional differential transform scheme

In this section, we set up an analytical framework, based on a new version of the differential transform method, for furnishing the solution of differential equations endowed with twofold Caputo-fractional derivatives. This new approach generalizes the traditional differential transform ideas and provides a powerful tool for analytically studying the combined effects of two distinct memory parameters.

Definition 2.1 ([41]). A bivariate (α,β)-fractional power series (simply denoted by (α,β)-FPS) around $(0,0)$ is a power series with two non-integer parameters $\alpha,\beta\in(0,1)$ in the form:

$$\sum_{\substack{n+m=0\\ n,m\in\mathbb{N}_*}}^{\infty} a_{nm}\,t^{n\alpha}x^{m\beta} = \underbrace{a_{00}}_{n+m=0} + \underbrace{a_{10}\,t^{\alpha} + a_{01}\,x^{\beta}}_{n+m=1} + \underbrace{a_{20}\,t^{2\alpha} + a_{11}\,t^{\alpha}x^{\beta} + a_{02}\,x^{2\beta}}_{n+m=2} + \cdots, \quad (3)$$

with $(t,x)\geq(0,0)$ are variables, and a_{nm}'s are the series coefficients.

Notation 1. We will simply write $\Gamma(n\alpha+1)$ as $\Gamma_\alpha(n)$.

Lemma 2.2 ([41]). *Suppose $u(t,x)$ has an (α,β)-FPS representation for $(t,x)\in[0,\mathrm{T})\times[0,\mathrm{X})$ and $\mathcal{D}_t^{i\alpha}\mathcal{D}_x^{j\beta}\left[u(t,x)\right]\in\mathcal{C}\left((0,\mathrm{T})\times(0,\mathrm{X})\right)$ for all $i,j\in\mathbb{N}_*$. Then for $(t,x)\geq(0,0)$ we have*

$$\begin{aligned}\mathcal{D}_t^{i\alpha}\mathcal{D}_x^{j\beta}\left[u(t,x)\right] &= \sum_{\substack{n+m=i+j\\ n\geq i,\, m\geq j}}^{\infty} a_{nm}\frac{\Gamma_\alpha(n)\Gamma_\beta(m)}{\Gamma_\alpha(n-i)\Gamma_\beta(m-j)}t^{(n-i)\alpha}x^{(m-j)\beta}\\ &= \sum_{n+m=0}^{\infty} a_{n+i,m+j}\frac{\Gamma_\alpha(n+i)\Gamma_\beta(m+j)}{\Gamma_\alpha(n)\Gamma_\beta(m)}t^{n\alpha}x^{m\beta}.\end{aligned} \quad (4)$$

Moreover,

$$a_{nm} = \frac{\mathcal{D}_t^{n\alpha}\mathcal{D}_x^{m\beta}\left[u(0,0)\right]}{\Gamma_\alpha(n)\Gamma_\beta(m)}. \quad (5)$$

Definition 2.3. The $(n\alpha,m\beta)$-fractional differential transform (abbreviated by $(n\alpha,m\beta)$-FDT) $U_\alpha^\beta(n,m)$ of the function $u(t,x)$ at $(0,0)$ for $n,\ m\in\mathbb{N}_*$ is defined by:

$$U_\alpha^\beta(n,m) = \frac{\mathcal{D}_t^{n\alpha}\mathcal{D}_x^{m\beta}\left[u(0,0)\right]}{\Gamma_\alpha(n)\Gamma_\beta(m)}. \quad (6)$$

Definition 2.4. An (α,β)-fractional differential inverse transform of $\left\{U_\alpha^\beta(n,m)\right\}_{(n,m)\in\mathbb{N}_*^2}$ at $(0,0)$ is defined by:

$$u(t,x) = \sum_{n+m=0}^{\infty} U_\alpha^\beta(n,m)\,t^{n\alpha}x^{m\beta}. \quad (7)$$

The next result gives the $(n\alpha,m\beta)$-FDT for some fundamental functional operators.

Theorem 2.5. *Let $U_\alpha^\beta(n,m)$, $V_\alpha^\beta(n,m)$ and $W_\alpha^\beta(n,m)$ be the $(n\alpha,m\beta)$-FDT of the functions $u(t,x), v(t,x)$ and $w(x,t)$, respectively, at $(0,0)$. Then the following properties hold:*

(i) *If $u(t,x)=v(t,x)\pm c\,w(t,x)$, then $U_\alpha^\beta(n,m)=V_\alpha^\beta(n,m)\pm c\,W_\alpha^\beta(n,m)$.*

(ii) *If $u(t,x) = \mathcal{D}_t^{i\alpha}\mathcal{D}_x^{j\beta}\left[v(t,x)\right]$, then $U_\alpha^\beta(n,m) = \dfrac{\Gamma_\alpha(n+i)\Gamma_\beta(m+j)}{\Gamma_\alpha(n)\Gamma_\beta(m)}V_\alpha^\beta(n+i,m+j)$.*

(iii) *If $u(t,x)=v(t,x)\,w(t,x)$, then $U_\alpha^\beta(n,m) = \sum_{r=0}^{n}\sum_{k=0}^{m}V_\alpha^\beta(r,k)\,W_\alpha^\beta(n-r,m-k)$.*

Proof. (i) Follows immediately from the linearity of the Caputo differential operator. By virtue of Lemma 2.2, (ii) follows directly. For (iii),

$$\begin{aligned} &u(t,x)\\ &=\left(\sum_{n+m=0}^{\infty} V_{\alpha}^{\beta}(n,m)\, t^{n\alpha}x^{m\beta}\right)\left(\sum_{n+m=0}^{\infty} W_{\alpha}^{\beta}(n,m)\, t^{n\alpha}x^{m\beta}\right)\\ &=\sum_{n+m=0}^{\infty}\left(\sum_{r=0}^{n}\sum_{k=0}^{m} V_{\alpha}^{\beta}(r,k)\, W_{\alpha}^{\beta}(n-r,m-k)\right) t^{n\alpha}x^{m\beta} \end{aligned} \tag{8}$$

as desired. □

Consequently, some essential transformed functions at $(0,0)$ in fractal 2-dimensional space are listed in Table 1.

3 Interpretation aspects of $(n\alpha, m\beta)$-FDT

In this section, two significant applications of the proposed $(n\alpha, m\beta)$-FDT will be presented to illustrate its broad utility. The first one is about approximating the mixed fractional derivatives in Caputo's sense, and the second one is about extracting closed-form solution of temporal-spatial physical models in fractal 2-dimensional space ((α,β)-models).

3.1 Approximation of (mixed-higher) fractional derivatives

Let $v(t,x)$ be a non-constant bivariate real analytic function with Taylor's double series representation around $(0,0)$:

$$v(t,x) = \sum_{n+m=0}^{\infty} \frac{1}{n!m!}\frac{\partial^{(n+m)}v(0,0)}{\partial t^n \partial x^m}\, t^n x^m. \tag{9}$$

By setting $u(t,x) = v(t^{\alpha}, x^{\beta})$, we have,

$$u(t,x) = \sum_{n+m=0}^{\infty} \frac{1}{n!m!}\frac{\partial^{(n+m)}v(0,0)}{\partial t^n \partial x^m}\, t^{n\alpha} x^{m\beta}. \tag{10}$$

On the other hand,

$$u(t,x) = \sum_{n+m=0}^{\infty} U_{\alpha}^{\beta}(n,m)\, t^{n\alpha}x^{m\beta}. \tag{11}$$

Thus, by equating the coefficients of like powers in (10) and (11), we obtain:

$$U_{\alpha}^{\beta}(n,m) = \frac{1}{n!m!}\frac{\partial^{(n+m)}v(0,0)}{\partial t^n \partial x^m}. \tag{12}$$

In other words, under certain conditions, the $(n\alpha, m\beta)$-FDT (and hence the mixed-higher fractional derivatives) can be written in terms of mixed-higher integer partial derivatives. Now, from Theorem 2.5, we have,

$$\begin{aligned} &\mathcal{D}_t^{r\alpha}\,\mathcal{D}_x^{k\beta}\big[u(t,x)\big]\\ &=\sum_{n+m=0}^{\infty}\frac{\Gamma_\alpha(n+r)\Gamma_\beta(m+k)}{\Gamma_\alpha(n)\Gamma_\beta(m)}U_{\alpha}^{\beta}(n+r,m+k)t^{n\alpha}x^{m\beta}\\ &=\sum_{n+m=0}^{\infty}\frac{\Gamma_\alpha(n+r)\Gamma_\beta(m+k)}{\Gamma_\alpha(n)\Gamma_\beta(m)}\frac{\frac{\partial^{(n+r+m+k)}v(0,0)}{\partial t^{(n+r)}\partial x^{(m+k)}}}{(n+r)!(m+k)!}t^{n\alpha}x^{m\beta}\\ &=\sum_{\substack{n+m=r+k\\ n\ge r,\, m\ge k}}^{\infty}\frac{\frac{\partial^{(n+m)}v(0,0)}{\partial t^n\partial x^m}}{n!m!}\frac{\Gamma_\alpha(n)\Gamma_\beta(m)\, t^{(n-r)\alpha}x^{(m-k)\beta}}{\Gamma_\alpha(n-r)\Gamma_\beta(m-k)}, \end{aligned} \tag{13}$$

and thus, the (mixed-higher) fractional derivatives of $u(t,x)$ can be approximated by the Nth partial sum of (13).

Example 3.1.1. Consider the transcendental function $u(t,x) = \ln\left(1+t^{\alpha}x^{2\beta}\right)$, where $\alpha, \beta \in (0,1)$ and $t, x \geq 0$. Following the same notation and hypothesis as in the previous discussion,

$$v(t,x) = \ln\left(1+tx^2\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}t^n x^{2n}, \tag{14}$$

and

$$u(t,x) = \ln\left(1+t^{\alpha}x^{2\beta}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}t^{n\alpha} x^{2n\beta}. \tag{15}$$

This consequently implies that,

$$\mathcal{D}_t^{r\alpha}\,\mathcal{D}_x^{k\beta}\big[u(0,0)\big] = \begin{cases} \dfrac{(-1)^{r+1}\Gamma_\alpha(r)\Gamma_\beta(k)}{r}, & r\geq 1\ \&\ m=2r\\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

and the Nth partial sum approximation of the mixed-higher fractional derivatives of $u(t,x)$ is:

$$\begin{aligned} &\mathcal{D}_t^{r\alpha}\,\mathcal{D}_x^{k\beta}\big[u(t,x)\big]\\ &\approx\sum_{n=r}^{N}\frac{(-1)^{n+1}\Gamma_\alpha(n)\Gamma_\beta(2n)}{n\,\Gamma_\alpha(n-r)\Gamma_\beta(2n-r)}t^{(n-r)\alpha}x^{(2n-r)\beta}. \end{aligned} \tag{17}$$

Tables 2, 3, and 4 show the approximate values of $D_t^{\alpha}\left[u(t,x)\right]$, $D_x^{\beta}\left[u(t,x)\right]$, and $D_t^{\alpha}D_x^{\beta}\left[u(t,x)\right]$, respectively, at different mesh points $(t,x) \in (0,1)\times(0,1)$ and different values of $\alpha, \beta \in (0,1)$ using the 12th and 18th partial sum approximations (17). In each table, the last column represents the absolute error of the approximate values against the exact values when $\alpha, \beta \to 1$. It is evident that we have high accuracy when $\alpha, \beta \to 1$, which indicates that the other approximate values are likely to be legitimate. On the other hand, it's clear that the accuracy of the proposed

Table 1: The $(n\alpha, m\beta)$-FDT of some essential functions at (0, 0).

Function $u(t, x)$	Transformed form
$t^{r\alpha}x^{s\beta}$	$U_\alpha^\beta(n,m)=\begin{cases}1; & n=r \text{ and } m=s\\ 0; & \text{otherwise.}\end{cases}$
$E_\beta\left(\lambda x^\beta\right)=\sum_{m=0}^{\infty}\frac{\lambda^m x^{m\beta}}{\Gamma_\beta(m)}$	$U_\alpha^\beta(n,m)=\begin{cases}\frac{\lambda^m}{\Gamma_\beta(m)}; & n=0,\ m\geq 0\\ 0; & \text{otherwise.}\end{cases}$
$t^{r\alpha}x^{s\beta}E_\beta\left(\lambda x^\beta\right)$	$U_\alpha^\beta(n,m)=\begin{cases}\frac{\lambda^{m-s}}{\Gamma_\beta(m-s)}; & n=r,\ m\geq s\\ 0; & \text{otherwise.}\end{cases}$
$E_\alpha\left(\mu t^\alpha\right)E_\beta\left(\lambda x^\beta\right)$	$U_\alpha^\beta(n,m)=\frac{\mu^n\lambda^m}{\Gamma_\alpha(n)\Gamma_\beta(m)}.$
$E_\beta\left(\mu x^\beta\right)E_\beta\left(\lambda x^\beta\right)$	$U_\alpha^\beta(n,m)=\begin{cases}\sum_{k=0}^{m}\frac{\mu^k\lambda^{m-k}}{\Gamma_\beta(k)\Gamma_\beta(m-k)}; & n=0,\ m\geq 0\\ 0; & \text{otherwise.}\end{cases}$
$\cos_\beta\left(\lambda x^\beta\right)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\lambda x^\beta\right)^{2m}}{\Gamma_\beta(2m)}$	$U_\alpha^\beta(n,m)=\begin{cases}\frac{(-1)^i\lambda^{2i}}{\Gamma_\beta(2i)}; & n=0,\ m=2i\\ 0; & \text{otherwise.}\end{cases}$
$t^{r\alpha}x^{s\beta}\cos_\beta\left(\lambda x^\beta\right)$	$U_\alpha^\beta(n,m)=\begin{cases}\frac{(-1)^i\lambda^{2i}}{\Gamma_\beta(2i)}; & n=r,\ m=2i+s\\ 0; & \text{otherwise.}\end{cases}$
$E_\alpha\left(\mu t^\alpha\right)E_\beta\left(\lambda x^\beta\right)\cos_\beta\left(\xi x^\beta\right)$	$U_\alpha^\beta(n,m)=\begin{cases}\sum_{r=0}^{n}\sum_{k=0}^{m}\frac{(-1)^i\mu^r\lambda^k\xi^{2i}}{\Gamma_\alpha(r)\Gamma_\beta(k)\Gamma_\beta(2i)}; & m-k=2i\\ 0; & \text{otherwise.}\end{cases}$
$\sin_\beta\left(\lambda x^\beta\right)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\lambda x^\beta\right)^{2m+1}}{\Gamma_\beta(2m+1)}$	$U_\alpha^\beta(n,m)=\begin{cases}\frac{(-1)^i\lambda^{2i+1}}{\Gamma_\beta(2i+1)}; & n=0,\ m=2i+1\\ 0; & \text{otherwise.}\end{cases}$
$t^{r\alpha}x^{s\beta}\sin_\beta\left(\lambda x^\beta\right)$	$U_\alpha^\beta(n,m)=\begin{cases}\frac{(-1)^i\lambda^{2i+1}}{\Gamma_\beta(2i+1)}; & n=r,\ m=(2i+1)+s\\ 0; & \text{otherwise.}\end{cases}$
$E_\alpha\left(\mu t^\alpha\right)E_\beta\left(\lambda x^\beta\right)\sin_\beta\left(\xi x^\beta\right)$	$U_\alpha^\beta(n,m)=\begin{cases}\sum_{r=0}^{n}\sum_{k=0}^{m}\frac{(-1)^i\mu^r\lambda^k\xi^{2i+1}}{\Gamma_\alpha(r)\Gamma_\beta(k)\Gamma_\beta(2i+1)}; & m-k=2i+1\\ 0; & \text{otherwise.}\end{cases}$

Table 2: The 12th and 18th partial sum approximations of $D_t^{\alpha}\left[\ln\left(1+t^{\alpha}x^{2\beta}\right)\right]$.

	$\alpha = 0.25, \beta = 0.5$		$\alpha = .5, \beta = 0.75$		$\alpha = .75, \beta \to 1$		$\alpha \to 1, \beta \to 1$		Absolute error	
(t, x)	12th	18th	12th	18th	12th	18th	12th	18th	12th	18th
(0.2, 0.3)	0.246106	0.246106	0.139180	0.139180	0.081004	0.081004	0.088408	0.088408	1.387×10^{-17}	0.
(0.2, 0.6)	0.451618	0.451619	0.364951	0.364951	0.305259	0.305259	0.335821	0.335821	6.494×10^{-15}	0.
(0.2, 0.9)	0.627805	0.627938	0.614507	0.614508	0.627397	0.627397	0.697074	0.697074	2.277×10^{-10}	4.218×10^{-15}
(0.5, 0.3)	0.240439	0.240439	0.135746	0.135746	0.079389	0.079389	0.086124	0.086124	2.775×10^{-17}	1.387×10^{-17}
(0.5, 0.6)	0.433826	0.433838	0.343136	0.343136	0.283946	0.283946	0.305085	0.305085	3.529×10^{-10}	1.199×10^{-14}
(0.5, 0.9)	0.593525	0.595222	0.557255	0.557498	0.545667	0.545703	0.576501	0.576512	1.122×10^{-05}	4.954×10^{-08}
(0.8, 0.3)	0.237148	0.237148	0.133396	0.133396	0.078072	0.078072	0.083955	0.083955	1.623×10^{-15}	0.
(0.8, 0.6)	0.423805	0.423850	0.329265	0.329267	0.268265	0.268265	0.279503	0.279503	9.101×10^{-08}	5.193×10^{-11}
(0.8, 0.9)	0.570037	0.575552	0.519780	0.523025	0.490452	0.492467	0.488811	0.491305	2.694×10^{-03}	1.994×10^{-04}

Table 3: The 12th and 18th partial sum approximations of $D_x^{\beta}\left[\ln\left(1+t^{\alpha}x^{2\beta}\right)\right]$.

	$\alpha = 0.25, \beta = 0.5$		$\alpha = .5, \beta = 0.75$		$\alpha = .75, \beta \to 1$		$\alpha \to 1, \beta \to 1$		Absolute error	
(t, x)	12th	18th	12th	18th	12th	18th	12th	18th	12th	18th
(0.2, 0.3)	0.365605	0.365605	0.247496	0.247496	0.174739	0.174739	0.117878	0.117878	0.	0.
(0.2, 0.6)	0.465594	0.465596	0.378034	0.378034	0.324000	0.324000	0.223881	0.223881	4.357×10^{-15}	0.
(0.2, 0.9)	0.520055	0.520300	0.458910	0.458912	0.433348	0.433348	0.309811	0.309811	1.012×10^{-10}	1.887×10^{-15}
(0.5, 0.3)	0.446824	0.446824	0.379072	0.379072	0.338640	0.338640	0.287081	0.287081	5.551×10^{-17}	5.551×10^{-17}
(0.5, 0.6)	0.557648	0.557681	0.552859	0.552859	0.587719	0.587719	0.508475	0.508475	5.882×10^{-10}	2.009×10^{-14}
(0.5, 0.9)	0.609320	0.613180	0.641301	0.641961	0.722259	0.722370	0.640557	0.640569	1.247×10^{-05}	5.504×10^{-08}
(0.8, 0.3)	0.494163	0.494163	0.468963	0.468963	0.471632	0.471632	0.447761	0.447761	8.715×10^{-15}	0.
(0.8, 0.6)	0.609691	0.609835	0.663831	0.663839	0.778120	0.778121	0.745341	0.745342	2.426×10^{-07}	1.384×10^{-10}
(0.8, 0.9)	0.645860	0.659608	0.737491	0.748431	0.893861	0.902533	0.868997	0.873432	4.789×10^{-03}	3.546×10^{-04}

Table 4: The 12th and 18th partial sum approximations of $D_t^{\alpha}D_x^{\beta}\left[\ln\left(1+t^{\alpha}x^{2\beta}\right)\right]$.

	$\alpha = 0.25, \beta = 0.5$		$\alpha = .5, \beta = 0.75$		$\alpha = .75, \beta \to 1$		$\alpha \to 1, \beta \to 1$		Absolute error	
(t, x)	12th	18th	12th	18th	12th	18th	12th	18th	12th	18th
(0.2, 0.3)	0.491034	0.491034	0.482897	0.482897	0.528885	0.528885	0.578970	0.578970	0.	0.
(0.2, 0.6)	0.620952	0.620956	0.719532	0.719532	0.939549	0.939549	1.044220	1.04422	2.808×10^{-13}	2.220×10^{-16}
(0.2, 0.9)	0.689533	0.690033	0.850421	0.850433	1.179360	1.17936	1.333090	1.33309	6.508×10^{-09}	1.729×10^{-13}
(0.5, 0.3)	0.476315	0.476315	0.463889	0.463889	0.508087	0.508087	0.549438	0.549438	5.551×10^{-16}	1.110×10^{-16}
(0.5, 0.6)	0.589654	0.589708	0.652931	0.652933	0.814671	0.814672	0.861821	0.861821	1.511×10^{-08}	7.541×10^{-13}
(0.5, 0.9)	0.637697	0.643886	0.729390	0.731760	0.899014	0.900031	0.911525	0.911840	3.171×10^{-04}	2.060×10^{-06}
(0.8, 0.3)	0.467845	0.467845	0.451109	0.451109	0.491465	0.491465	0.522110	0.522110	1.404×10^{-13}	1.110×10^{-16}
(0.8, 0.6)	0.572191	0.572399	0.612230	0.612254	0.729095	0.729099	0.723348	0.723352	3.876×10^{-06}	3.250×10^{-09}
(0.8, 0.9)	0.592000	0.611084	0.627387	0.657253	0.678619	0.732261	0.587285	0.654515	7.547×10^{-02}	8.247×10^{-03}

approximation is lower when the values of t and x are far from the origin (center of the expansion). However, comparing the absolute error of the 18th approximation with the 12th approximation suggests that adding up more terms of the series expansion will instantly lead to more accuracy. This suggests that this expansion can be used as a routine technique to estimate (mixed-higher) fractional derivatives.

3.2 Analytic solution of (α, β)-physical models

Here, we consider several hybrid (non)linear (α, β)-physical models and furnish their solutions in closed-forms by employing the above proposed idea of $(n\alpha, m\beta)$-FDT (we will call it by (α, β)-FDTM). For validation and comparison purposes, we choose to recall some of these models that have been recently solved by a modified version of the power series method [41]. In all our illustrative models, we assume that $\alpha,\ \beta \in (0,1)$ and $t,\ x \geq 0$.

Example 3.2.1. Consider the following wave model in fractal 2-dimensional space:

$$\mathcal{D}_t^{2\alpha}\left[u(t,x)\right] = c^2\,\mathcal{D}_x^{2\beta}\left[u(t,x)\right], \tag{18}$$

based on the initial conditions:

$$u(0,x) = E_\beta\left(x^\beta\right) \text{ and } \mathcal{D}_t^{\alpha}\left[u(0,x)\right] = 0. \tag{19}$$

We search for a solution to (18)–(19) in the form:

$$u(t,x) = \sum_{n+m=0}^{\infty} U_\alpha^\beta(n,m)\,t^{n\alpha}x^{m\beta}, \tag{20}$$

where $U_\alpha^\beta(n,m)$ is the $(n\alpha, m\beta)$-FDT of $u(t,x)$. Upon applying the $(n\alpha, m\beta)$-FDT and using the related properties in Theorem 2.5, we obtain the following $(n\alpha, m\beta)$-transformed form for eqs. (18) and (19), respectively.

$$\frac{\Gamma_\alpha(n+2)}{\Gamma_\alpha(n)}\,U_\alpha^\beta(n+2,m) = c^2\,\frac{\Gamma_\beta(m+2)}{\Gamma_\beta(m)}\,U_\alpha^\beta(n,m+2), \tag{21}$$

subject to,

$$U_\alpha^\beta(n,m) = \begin{cases} \dfrac{1}{\Gamma_\beta(m)}; & \text{for } n=0,\ m\geq 0 \\ 0; & \text{for } n=1,\ m\geq 0. \end{cases} \tag{22}$$

Solving recursively eq. (21) for $U_\alpha^\beta(n,m)$, while taking into account equation (22), we obtain:

$$U_\alpha^\beta(n,m) = \begin{cases} \dfrac{c^n}{\Gamma_\alpha(n)\Gamma_\beta(m)}; & \text{for } n \text{ is even, } m\geq 0 \\ 0; & \text{for } n \text{ is odd, } m\geq 0. \end{cases} \tag{23}$$

Plugging (23) back into (20) confers the following memory solution of the wave model (18)–(19):

$$\begin{aligned} u(t,x) &= \sum_{n+m=0}^{\infty} \frac{c^{2n}}{\Gamma_\alpha(2n)\Gamma_\beta(m)}\,t^{2n\alpha}x^{m\beta} \\ &= \left(\sum_{n=0}^{\infty}\frac{\left(ct^\alpha\right)^{2n}}{\Gamma_\alpha(2n)}\right)\left(\sum_{m=0}^{\infty}\frac{x^{m\beta}}{\Gamma_\beta(m)}\right) \\ &= \cosh_\alpha\left(ct^\alpha\right)\,E_\beta\left(x^\beta\right). \end{aligned} \tag{24}$$

We remark here that, with $\alpha = \beta$, the resulting solution (24) reduced to $u(t,x) = \cosh_\alpha(ct^\alpha)\,E_\alpha(x^\alpha)$, which is in complete agreement with what was obtained by using the fractional Laplace variation iteration method [51]. Moreover, if $\alpha, \beta \to 1$, we get the closed-form solution $u(t,x) = e^x\cosh(ct)$ for the traditional wave model of (18)–(19).

Example 3.2.2. Consider the following model with mixed fractional derivatives in 2-dimensional space:

$$\mathcal{D}_t^{\alpha}\left[u(t,x)\right] = \mathcal{D}_t^{\alpha}\mathcal{D}_x^{\beta}\left[u(t,x)\right] - u(t,x), \tag{25}$$

based on the following conditions,

$$u(0,x) = E_\beta\left(2x^\beta\right) \text{ and } u(t,0) = E_\alpha\left(t^\alpha\right). \tag{26}$$

By applying the $(n\alpha, m\beta)$-FDT and using the related properties in Theorem 2.5, we obtain the following $(n\alpha, m\beta)$-transformed form for eqs. (25) and (26), respectively.

$$\begin{aligned} &\frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}U_\alpha^\beta(n+1,m) + U_\alpha^\beta(n,m) \\ &\quad - \frac{\Gamma((n+1)\alpha+1)\Gamma((m+1)\beta+1)}{\Gamma(n\alpha+1)\Gamma(m\beta+1)}U_\alpha^\beta(n+1,m+1) = 0, \end{aligned} \tag{27}$$

subject to,

$$U_\alpha^\beta(n,m) = \begin{cases} \dfrac{1}{\Gamma(n\alpha+1)}; & \text{for } n\geq 0,\ m=0 \\ \dfrac{2^m}{\Gamma(m\beta+1)}; & \text{for } n=0,\ m\geq 0. \end{cases} \tag{28}$$

Solving recursively eq. (27) for $U_\alpha^\beta(n,m)$, while taking into account eq. (28), we obtain:

$$U_\alpha^\beta(n,m) = \frac{2^m}{\Gamma(n\alpha+1)\Gamma(m\beta+1)}. \tag{29}$$

Plugging (29) back into (20) confers the following memory solution of the model (25)–(26):

$$\begin{aligned} u(t,x) &= \sum_{n+m=0}^{\infty} \frac{2^m}{\Gamma(n\alpha+1)\Gamma(m\beta+1)}\,t^{n\alpha}x^{m\beta} \\ &= \left(\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right)\left(\sum_{m=0}^{\infty}\frac{2^m\,x^{m\beta}}{\Gamma(m\beta+1)}\right) \\ &= E_\alpha\left(t^\alpha\right)\,E_\beta\left(2x^\beta\right). \end{aligned} \tag{30}$$

We note here that, when $\alpha, \beta \to 1$, we get the closed-form solution $u(t,x) = e^{t+2x}$ for the integer-version of the model (25)–(26).

Figures 1 and 2 display the level-curves of the 12th approximate solutions, $u_{12}(t,x) = \sum_{n+m=0}^{12} \frac{2^m t^{n\alpha} x^{m\beta}}{\Gamma(n\alpha+1)\Gamma(m\beta+1)}$ of example 3.2.2 tagged with the fractional parameters α and β, respectively. Visibly, the curves are continuously animating as $\alpha \to 1$ (or $\beta \to 1$) to reach the associated one of the integer-order case. This phenomenon probably fosters the potential meaning of the fractional derivative as a memory reflexive.

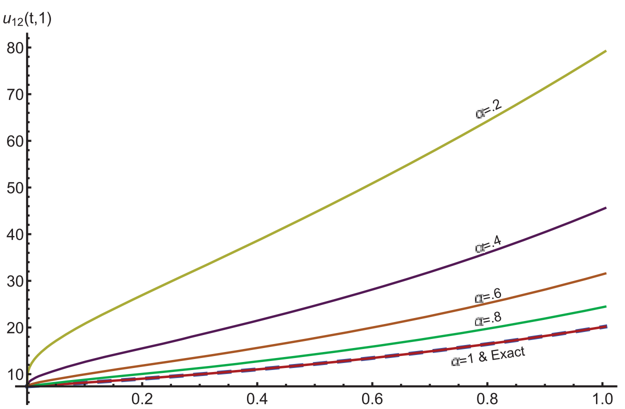

Figure 1: Level-curves $u_{12}(t,1)$ at distinct values of $\alpha \in (0,1]$, $\beta = 1$, and $t \in [0,1]$.

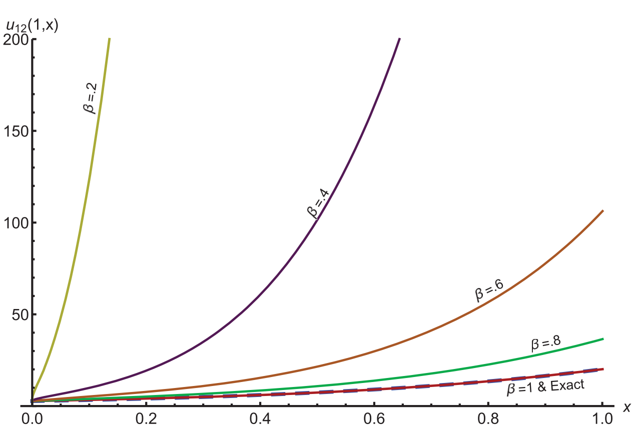

Figure 2: Level-curves $u_{12}(1,x)$ at distinct values of $\beta \in (0,1]$, $\alpha = 1$, and $x \in [0,1]$.

Example 3.2.3. Consider the following Burgers' model in fractal 2-dimensional space:

$$\mathcal{D}_t^{\alpha}\left[u(t,x)\right] + \mathcal{D}_x^{\beta}\left[u(t,x)\right] - \mathcal{D}_x^{2\beta}\left[u(t,x)\right] = 0, \tag{31}$$

based on the initial condition,

$$u(0,x) = E_{\beta}\left(-x^{\beta}\right). \tag{32}$$

By applying the $(n\alpha, m\beta)$-FDT and utilizing the associated properties in Theorem 2.5, we get the following $(n\alpha, m\beta)$-transformed form of eqs. (31) and (32), respectively.

$$\begin{aligned}&\frac{\Gamma_\alpha(n+1)}{\Gamma_\alpha(n)} U_\alpha^\beta(n+1,m) + \frac{\Gamma_\beta(m+1)}{\Gamma_\beta(m)} U_\alpha^\beta(n,m+1)\\ &\quad - \frac{\Gamma_\beta(m+2)}{\Gamma_\beta(m)} U_\alpha^\beta(n,m+2) = 0,\end{aligned} \tag{33}$$

subject to,

$$U_\alpha^\beta(0,m) = \frac{(-1)^m}{\Gamma_\beta(m).} \tag{34}$$

By solving recursively eq. (33) and using the transformed initial condition (34), we obtain,

$$U_\alpha^\beta(n,m) = \frac{(-1)^m 2^n}{\Gamma_\alpha(n)\Gamma_\beta(m)}. \tag{35}$$

Plugging (35) back into (20) confers the following memory solution of the Burgers' model (31)–(32):

$$\begin{aligned}u(t,x) &= \sum_{n+m=0}^{\infty} \frac{(-1)^m 2^n}{\Gamma_\alpha(n)\Gamma_\beta(m)} t^{n\alpha} x^{m\beta}\\ &= \left(\sum_{n=0}^{\infty} \frac{(2t^\alpha)^n}{\Gamma_\alpha(n)}\right)\left(\sum_{m=0}^{\infty} \frac{(-x^\beta)^m}{\Gamma_\beta(m)}\right)\\ &= E_\alpha(2t^\alpha)E_\beta(-x^\beta).\end{aligned} \tag{36}$$

We remark here that, when $\beta \to 1$, the same solution was obtained by using the homotopy analysis method [52]. Moreover, if $\alpha, \beta \to 1$, we attain the closed-form solution $u(t,x) = e^{2t-x}$ for the traditional Burgers' model of (31)–(32).

Example 3.2.4. Consider the following telegraph model in fractal 2-dimensional space:

$$\mathcal{D}_x^{2\beta}\left[u(t,x)\right] = \mathcal{D}_t^{2\alpha}\left[u(t,x)\right] + \mathcal{D}_t^{\alpha}\left[u(t,x)\right] + u(t,x), \tag{37}$$

based on the initial conditions,

$$u(t,0) = E_\alpha\left(-t^\alpha\right) \quad \text{and} \quad \mathcal{D}_x^\beta\left[u(t,0)\right] = E_\alpha\left(-t^\alpha\right). \tag{38}$$

Again, by performing the $(n\alpha, m\beta)$-FDT and utilizing Theorem 2.5, the following $(n\alpha, m\beta)$-transformed form of eqs. (37) and (38), respectively, are obtained:

$$\frac{\Gamma_\beta(m+2)}{\Gamma_\beta(m)} U_\alpha^\beta(n,m+2) = \frac{\Gamma_\alpha(n+2)}{\Gamma_\alpha(n)} U_\alpha^\beta(n+2,m) + \frac{\Gamma_\alpha(n+1)}{\Gamma_\alpha(n)} U_\alpha^\beta(n+1,m) + U_\alpha^\beta(n,m), \tag{39}$$

and

$$U_\alpha^\beta(n,m) = \begin{cases} \frac{(-1)^n}{\Gamma_\alpha(n)}; & \text{for } n \ge 0,\ m=0 \\ \frac{(-1)^n}{\Gamma_\alpha(n)\Gamma_\beta(1)}; & \text{for } n \ge 0,\ m=1. \end{cases} \tag{40}$$

Solving recursively eq. (39), and taking into consideration eq. (40), we obtain:

$$U_\alpha^\beta(n,m) = \frac{(-1)^n}{\Gamma_\alpha(n)\Gamma_\beta(m).} \tag{41}$$

Plugging (41) back into (20) confers the following memory solution of the telegraph model (37)–(38):

$$\begin{aligned} u(t,x) &= \sum_{n+m=0}^{\infty} \frac{(-1)^n}{\Gamma_\alpha(n)\Gamma_\beta(m)} t^{n\alpha} x^{m\beta} \\ &= \left(\sum_{n=0}^{\infty} \frac{(-t^\alpha)^n}{\Gamma_\alpha(n)}\right)\left(\sum_{m=0}^{\infty} \frac{x^{m\beta}}{\Gamma_\beta(m)}\right) \\ &= E_\alpha\left(-t^\alpha\right) E_\beta\left(x^\beta\right). \end{aligned} \tag{42}$$

We point out here that the same solution was recently acquired by using a modified version to the Taylor's power series method (TPSM) [41]. Also, if $\alpha,\beta \to 1$, we attain the closed-form solution $u(t,x) = e^{x-t}$ for the traditional telegraph model of (37)–(38) [53].

Example 3.2.5. Consider the following nonhomogeneous KdV model in fractal 2-dimensional space:

$$\mathcal{D}_t^\alpha[u(t,x)] + \mathcal{D}_x^\beta[u(t,x)] + \mathcal{D}_x^{3\beta}[u(t,x)] = \frac{2t^\alpha \cos_\beta\left(x^\beta\right)}{\Gamma_\alpha(1)}, \tag{43}$$

based on the homogeneous initial condition,

$$u(0,x) = 0. \tag{44}$$

Applying the $(n\alpha, m\beta)$-FDT and using Theorem 2.5 yield the following $(n\alpha, m\beta)$-transformed form of eqs. (43) and (44), respectively.

$$\frac{\Gamma_\alpha(n+1)}{\Gamma_\alpha(n)} U_\alpha^\beta(n+1,m) + \frac{\Gamma_\beta(m+1)}{\Gamma_\beta(m)} U_\alpha^\beta(n,m+1) + \frac{\Gamma_\beta(m+3)}{\Gamma_\beta(m)} U_\alpha^\beta(n,m+3) = \begin{cases} \frac{2(-1)^i}{\Gamma_\alpha(1)\Gamma_\beta(2i)}; & n=1\ \&\ m=2i \\ 0; & \text{o.w,} \end{cases} \tag{45}$$

with

$$U_\alpha^\beta(0,m) = 0, \quad \text{for } m \ge 0. \tag{46}$$

Consequently, eqs. (45) and (46) imply that,

$$U_\alpha^\beta(n,m) = \begin{cases} \frac{2(-1)^i}{\Gamma_\alpha(n)\Gamma_\beta(2i)}; & \text{for } n=2,\ m=2i \\ 0; & \text{otherwise} \end{cases}. \tag{47}$$

Plugging (47) back into (20) confers the following memory solution of the KdV model (43)–(44):

$$\begin{aligned} u(t,x) &= \sum_{m=0}^{\infty} \frac{2(-1)^m}{\Gamma_\alpha(2)\Gamma_\beta(2m)} t^{2\alpha} x^{2m\beta} \\ &= \frac{2t^{2\alpha}}{\Gamma_\alpha(2)} \cos_\beta\left(x^\beta\right), \end{aligned} \tag{48}$$

which is identical to the solution attained by using TPSM [41]. Moreover, when $\alpha,\beta \to 1$, the solution (48) reduced to $u(t,x) = t^2\cos(x)$, which is the closed-form solution for the traditional KdV model (43)–(44).

Example 3.2.6. Consider the nonlinear gas dynamics model in fractal 2-dimensional space:

$$\mathcal{D}_t^\alpha[u(t,x)] + u(t,x)\mathcal{D}_x^\beta[u(t,x)] - u(t,x)(1-u(t,x)) = 0, \tag{49}$$

with initial condition,

$$u(0,x) = E_\beta\left(-x^\beta\right). \tag{50}$$

By applying the $(n\alpha, m\beta)$-FDT and using the related properties in Theorem 2.5, we obtain the following $(n\alpha, m\beta)$-transformed form of eqs. (49) and (50), respectively.

$$\begin{aligned} &\frac{\Gamma_\alpha(n+1)}{\Gamma_\alpha(n)} U_\alpha^\beta(n+1,m) \\ &+ \sum_{r=0}^{n}\sum_{k=0}^{m} \frac{\Gamma_\beta(k+1)}{\Gamma_\beta(k)} U_\alpha^\beta(r,k+1) U_\alpha^\beta(n-r,m-k) \\ &+ \sum_{r=0}^{n}\sum_{k=0}^{m} U_\alpha^\beta(r,k) U_\alpha^\beta(n-r,m-k) - U_\alpha^\beta(n,m) = 0, \end{aligned} \tag{51}$$

subject to,

$$U_\alpha^\beta(0,m)=\frac{(-1)^m}{\Gamma_\beta(m)},\quad \text{for } m\geq 0. \tag{52}$$

Solving recursively eq. (51), while taking into account eq. (52), leads to

$$U_\alpha^\beta(n,m)=\frac{(-1)^m}{\Gamma_\alpha(n)\Gamma_\beta(m)}. \tag{53}$$

Plugging (53) back into (20) confers the following memory solution of the gas model (49)–(50).

$$\begin{aligned} u(t,x) &= \sum_{n+m=0}^{\infty}\frac{(-1)^m}{\Gamma_\alpha(n)\Gamma_\beta(m)}t^{n\alpha}x^{m\beta} \\ &= \left(\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma_\alpha(n)}\right)\left(\sum_{m=0}^{\infty}\frac{\left(-x^\beta\right)^m}{\Gamma_\beta(m)}\right) \\ &= E_\alpha\left(t^\alpha\right)E_\beta\left(-x^\beta\right), \end{aligned} \tag{54}$$

which is identical to the solution attained by using TPSM [41]. If $\beta\to 1$, we have the same solution acquired by the fractional differential transform method [54]. Further, when $\alpha,\beta\to 1$, the solution (54) reduced to $u(t,x)=e^{t-x}$, which is the closed-form solution for the traditional gas model of (49)–(50) [54–57].

Example 3.2.7. Consider the nonlinear nonhomogeneous gas dynamical model in fractal 2D space:

$$\begin{aligned} &\mathcal{D}_t^\alpha\left[u(t,x)\right]+u(t,x)\mathcal{D}_x^\beta\left[u(t,x)\right]-u(t,x)\left(1-u(t,x)\right) \\ &=-E_\alpha\left(t^\alpha\right)E_\beta\left(-x^\beta\right), \end{aligned} \tag{55}$$

with initial condition,

$$u(0,x)=1-E_\beta\left(-x^\beta\right). \tag{56}$$

By performing the $(n\alpha,m\beta)$-FDT and utilizing the related properties in Theorem 2.5, we obtain the following $(n\alpha, m\beta)$-transformed form of eqs. (55) and (56), respectively:

$$\begin{aligned} &\frac{\Gamma_\alpha(n+1)}{\Gamma_\alpha(n)}U_\alpha^\beta(n+1,m) \\ &+\sum_{r=0}^{n}\sum_{k=0}^{m}\frac{\Gamma_\beta(k+1)}{\Gamma_\beta(k)}U_\alpha^\beta(r,k+1)U_\alpha^\beta(n-r,m-k) \\ &+\sum_{r=0}^{n}\sum_{k=0}^{m}U_\alpha^\beta(r,k)U_\alpha^\beta(n-r,m-k)-U_\alpha^\beta(n,m) \\ &=\frac{(-1)^{m+1}}{\Gamma_\alpha(n)\Gamma_\beta(m)}, \end{aligned} \tag{57}$$

and

$$U_\alpha^\beta(0,m)=\begin{cases} 0; & \text{for } m=0 \\ \dfrac{(-1)^{m+1}}{\Gamma_\beta(m)}; & \text{for } m\geq 1. \end{cases} \tag{58}$$

Solving recursively eq. (57), while taking into account eq. (58), implies that:

$$U_\alpha^\beta(n,m)=\begin{cases} 0; & \text{for } n=m=0 \\ \dfrac{(-1)^{m+1}}{\Gamma_\alpha(n)\Gamma_\beta(m)}; & \text{otherwise.} \end{cases} \tag{59}$$

Plugging (59) back into (20) confers the following memory solution of the gas model (55)–(56).

$$\begin{aligned} u(t,x) &= \sum_{n+m=1}^{\infty}\frac{(-1)^{m+1}}{\Gamma_\alpha(n)\Gamma_\beta(m)}t^{n\alpha}x^{m\beta} \\ &= 1-\left(\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma_\alpha(n)}\right)\left(\sum_{m=0}^{\infty}\frac{\left(-x^\beta\right)^m}{\Gamma_\beta(m)}\right) \\ &= 1-E_\alpha\left(t^\alpha\right)E_\beta\left(-x^\beta\right), \end{aligned} \tag{60}$$

which is identical to the solution obtained by using TPSM [41]. If $\beta\to 1$, we have the same solution acquired by the fractional differential transform method [54]. Moreover, when $\alpha,\beta\to 1$, the solution (60) reduces to $u(t,x)=1-e^{t-x}$ which is the closed-form solution for the traditional gas model of (55)–(56) [57].

Example 3.2.8. Finally, consider the nonlinear convection-diffusion model in fractal 2-dimensional space:

$$\begin{aligned} \mathcal{D}_t^\alpha\left[u(t,x)\right] &= \mathcal{D}_x^{2\beta}\left[u(t,x)\right]-\mathcal{D}_x^\beta\left[u(t,x)\right] \\ &\quad +u(t,x)\mathcal{D}_x^{2\beta}\left[u(t,x)\right]+u(t,x)\left(1-u(t,x)\right), \end{aligned} \tag{61}$$

based on the nonhomogeneous initial condition:

$$u(0,x)=E_\beta\left(x^\beta\right). \tag{62}$$

By applying the $(n\alpha,m\beta)$-FDT and using the related properties in Theorem 2.5, we obtain the following $(n\alpha, m\beta)$-transformed form of eqs. (61) and (62), respectively:

$$\frac{\Gamma_\alpha(n+1)}{\Gamma_\alpha(n)}U_\alpha^\beta(n+1,m)=\frac{\Gamma_\beta(m+2)}{\Gamma_\beta(m)}U_\alpha^\beta(n,m+2)$$

$$
\begin{aligned}
&-\frac{\Gamma_\beta(m+1)}{\Gamma_\beta(m)}U_\alpha^\beta(n,m+1)\\
&+\sum_{r=0}^{n}\sum_{k=0}^{m}\frac{\Gamma_\beta(k+2)}{\Gamma_\beta(k)}U_\alpha^\beta(r,k+2)U_\alpha^\beta(n-r,m-k) \quad (63)\\
&-\sum_{r=0}^{n}\sum_{k=0}^{m}U_\alpha^\beta(r,k)U_\alpha^\beta(n-r,m-k)+U_\alpha^\beta(n,m),
\end{aligned}
$$

with

$$U_\alpha^\beta(0,m)=\frac{1}{\Gamma_\beta(m)}, \quad \text{for } m\geq 0. \quad (64)$$

Solving recursively eq. (63), while taking into account equation (64), we obtain:

$$U_\alpha^\beta(n,m)=\frac{1}{\Gamma_\alpha(n)\Gamma_\beta(m)}. \quad (65)$$

Substituting (65) back into (20) confers the following memory solution for the convection-diffusion model (61)–(62):

$$
\begin{aligned}
u(t,x) &= \sum_{n+m=1}^{\infty}\frac{1}{\Gamma_\alpha(n)\Gamma_\beta(m)}t^{n\alpha}x^{m\beta}\\
&=\left(\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma_\alpha(n)}\right)\left(\sum_{m=0}^{\infty}\frac{x^{m\beta}}{\Gamma_\beta(m)}\right) \quad (66)\\
&= E_\alpha\left(t^\alpha\right)E_\beta\left(x^\beta\right).
\end{aligned}
$$

Particularly, if $\beta \to 1$, we acquire the same solution obtained by the Adomian decomposition method [6] and the He's homotopy perturbation method [7]. Moreover, when $\alpha,\beta \to 1$, we get the closed-form solution $u(t,x) = e^{t+x}$ for the traditional convection-diffusion model of (61)–(62) [58].

4 Conclusions

The main objective of this study is to broaden the differential transform idea to provide an analytic framework for the $(1+1)$-physical models endowed with both time and space memory parameters. To achieve this, a new representation of fractional power series is integrated with the differential transform method to transfer the functional operator under consideration into a recurrence equation. Validation of our approach has been carried out on several (α,β)-physical models, and the obtained results were pragmatic. Also, an estimate for higher order fractional derivatives of a bivariate function is given in Caputo's sense. We conclude our study with the following:

1. The scheme developed in this study is pioneering and attempts to provide an independent analytical treatment for (α,β)-models without using fractional transformation, linearization, or perturbation. This reveals a great prospective application of the suggested scheme as well as the generality of these (α,β)-models.
2. Compared to the recently developed (α,β)-Taylor's power series method [41], the new scheme demands less computational complexity because it directly transforms the (α,β)-model to a recurrence formula that is relatively easy to solve in a recursive manner.

The method proposed in this study introduces a considerable refinement in solving (α,β)-models analytically. Thus, research in this direction appears worth pursuing.

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