IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Date of publication xxxx xx, xxxx, date of current version xxxx xx, xxxx.

Digital Object Identifier xx.xxxx/ACCESS.2021.DOI

A New Sufficient Criterion for the Stability of 2-D Discrete Systems

APOSTOLOS KANELLAKIS[1], AND AYMAN TAWFIK[2]
[1]School of Electrical and Computer Engineering, National Technical University of Athens, 157 80 Athens, Greece, (e-mail: aik@mail.ntua.gr)
[2]Department of the Electrical and Computer Engineering, College of Engineering and Information Technology, Ajman University, P. O. Box: 346 Ajman, United Arab Emirates (e-mail: a.tawfik@ajman.ac.ae)

Corresponding author: Ayman Tawfik (e-mail: a.tawfik@ajman.ac.ae).

ABSTRACT During the past few decades, two and higher dimensional systems have been extensively applied in many areas of research. The representation of the 2-D systems in the frequency domain is usually given by its transfer function. The bounded-input bounded-output (BIBO) stability of the two dimensional discrete systems depends on the zeros of the characteristic polynomial which is the denominator of this transfer function. In this paper, a new sufficient criterion for the stability of two-dimensional linear shift-invariant discrete systems is presented. The new criterion is based on the sufficient condition for stable polynomials with complex coefficients and the stability criterion for 2-D discrete systems proposed by Murray and Delsarte et al. The new criterion is non-conservative for the stability testing of 2-D discrete systems. It is shown that the proposed sufficient criterion is simple enough to be applied for the stability checking of the 2-D discrete systems. The utility of the proposed criterion is demonstrated by examples.

INDEX TERMS 2-D Discrete systems, Transfer function, Polynomials, Stability, Sufficient condition

I. INTRODUCTION

DURING the past few decades, two and higher dimensional systems have been extensively applied in many areas of the study of broadband beamforming, digital filtering, image processing, multipass processes, gas filtration, thermal processes, geophysics, medical electronics, video and lightfield processing, sensor networks, 2-D discrete control systems, and so on [1] - [11]. In these applications, the signals are functions of two or more variables. The designer of systems designed for such applications is interested in the checking the stability of them. Criteria which provide sufficient and necessary conditions for the stability of 2-D discrete systems have been developed.

The stability analysis of 2-D discrete systems was studied in the frequency domain leading to many well known criteria and tests [12], [13] - [29]. It is a heavy computational task to test whether a 2-D polynomial is devoid of zeros in the unit bidisc. Hence, it is useful to have at hand simple sufficient conditions for the checking the stability. Such contributions for the stability of 2-D discrete systems have been presented in [30], [31]. Simple relations between the coefficients of the 2-D polynomial have been presented in these contributions. By using the sufficient conditions in [30], [31], the stability checking of the 2-D discrete systems can be carried out rapidly. In this paper a new sufficient criterion for linear 2-D discrete systems will be proposed. This is based on the sufficient condition for stable polynomials with complex coefficients presented in [34], [35] and the stability criterion proposed by Murray [32] and Delsarte et al [33]. Simple relations between the coefficients of the 2-D polynomial will be presented in the proposed criterion. By using the proposed criterion the stability checking of the 2-D discrete systems can be carried out rapidly.

Notation: The closed unit bidisc is denoted by $\overline{U}^2$

$$\overline{U}^2 = \{(z_1, z_2) | |z_1| \leq 1, |z_2| \leq 1\}$$

the open unit bidisc is denoted by U^2

$$U^2 = \{(z_1, z_2) | |z_1| < 1, |z_2| < 1\}$$

and the boundary of the unit bidisc is denoted by T^2

$$T^2 = \{(z_1, z_2) | |z_1| = 1, |z_2| = 1\}$$

Similar denotes are given for the one-variable regions $\overline{U}$, U and T. $\mathbf{R}$ and $\mathbf{C}$ denote the set of real numbers and complex numbers respectively. The region in which a stable polynomial is allowed to have its roots is defined as the stability region of the polynomial.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2021.3078076, IEEE Access

II. PRELIMINARIES

A single input-single output linear shift-invariant 2-D discrete system is represented in frequency domain by its transfer function $H(z_1, z_2)$ which is a rational function

$$H(z_1, z_2) = \frac{B(z_1, z_2)}{A(z_1, z_2)} \tag{1}$$

where $B(z_1, z_2), A(z_1, z_2)$ are two variable polynomials of the variables z_1, z_2, where $z_1, z_2 \in \mathbf{C}$. Systems (1) are assumed to have no non-essential singularities of the second kind on the unit bidisc T^2. The case of having non-essential singularities has been considered in [36]. The 2-D characteristic polynomial $A(z_1, z_2)$ is given by:

$$A(z_1, z_2) = \sum_{i=0}^{n} \sum_{l=0}^{m} \alpha_{il} z_1^i z_2^l \tag{2}$$

where $\alpha_{il} \in \mathbf{R}$ are the constant real coefficients of the polynomial.

Criteria which provide sufficient and necessary conditions for the bounded-input bounded-output (BIBO) stability of 2-D discrete systems have been developed in [12]. They require that the characteristic polynomial is devoid of zeros in the unit bidisc. The condition for BIBO stability of 2-D discrete systems is given below.

Stability condition for 2-D discrete systems [12]

The linear 2-D discrete system (1) is BIBO stable if and only if

$$A(z_1, z_2) \neq 0 \tag{3}$$

in the closed unit bidisc $\overline{U}^2$.

Stability Criterion for 2-D discrete systems [32], [33]

The linear 2-D discrete system (1) is BIBO stable if and only if

$$A(z, ze^{j\omega}) \neq 0, |z| \leq 1, \omega \in [0, 2\pi] \tag{4}$$

In the above criterion, the 2-D stability problem has been reduced in a 1-D stability problem with a characteristic polynomial whose coefficients are functions of $e^{j\omega}$ for $\omega \in [0, 2\pi]$.

It should be mentioned that using the arguments in [32], [33] numerous other stability criteria can be derived from (4) by replacing one or both variables by a finite Blaschke product since this is precisely the class of functions analytic in U, continuous on U and having unit modulus on T. An introduction to finite Blaschke products is presented in [37].

The following lemma is important for the derivation of the new sufficient condition.

Lemma 1 [34], [35]: The polynomial $A(z) = z^n + \alpha_n z^{n-1} + \cdots + \alpha_2 z + \alpha_1$ *with complex coefficients has all its roots inside the unit disc if*

$$\sum_{i=1}^{n} |\alpha_i|^2 < \frac{1}{n} \tag{5}$$

where $|\alpha_i|$ is the absolute value of the coefficient α_i. The above lemma implies that the polynomial $A(z)$ is stable.

III. THE NEW SUFFICIENT CRITERION FOR THE STABILITY OF 2-D DISCRETE SYSTEMS

In the below theorem the new sufficient criterion for the stability of linear 2-D discrete systems will be presented. For the derivation of this new stability criterion, the criterion presented in [32], [33] will be combined with the results in [34], [35].

Theorem: The linear 2-D discrete system (1) is BIBO stable if

$$\sum_{k=1}^{n+m} b_k^2 < \frac{|\alpha_{00}|^2}{n+m} \tag{6}$$

with

$$\left.\begin{array}{l}
b_1 = (|\alpha_{10}| + |\alpha_{01}|) \\
b_2 = (|\alpha_{20}| + |\alpha_{02}| + |\alpha_{11}|) \\
b_3 = (|\alpha_{30}| + |\alpha_{03}| + |\alpha_{12}| + |\alpha_{21}|) \\
b_4 = (|\alpha_{40}| + |\alpha_{04}| + |\alpha_{13}| + |\alpha_{31}| + |\alpha_{22}|) \\
\qquad \vdots \\
b_{n+m-3} = \\
(|\alpha_{n-3,m}| + |\alpha_{n-2,m-1}| + |\alpha_{n-1,m-2}| + |\alpha_{n,m-3}|) \\
b_{n+m-2} = (|\alpha_{n-2,m}| + |\alpha_{n,m-2}| + |\alpha_{n-1,m-1}|) \\
b_{n+m-1} = (|\alpha_{n,m-1}| + |\alpha_{n-1,m}|) \\
b_{n+m} = |\alpha_{nm}|
\end{array}\right\} \tag{7}$$

Proof: It follows from (2) and (4) that the polynomial $A(z, ze^{j\omega})$ can be written as

$$A(z, ze^{j\omega}) = \sum_{i=0}^{n} \sum_{l=0}^{m} \alpha_{il} e^{jl\omega} z^{i+l} \tag{8}$$

The reciprocal polynomial $A_1(z)$ of the polynomial $A(z, ze^{j\omega})$ can be written as

$$A_1(z) = z^{n+m} A(z^{-1}, z^{-1}e^{j\omega}) \tag{9}$$

and we define the polynomial $A_2(z)$ as $A_2(z) = A_1(z)/\alpha_{00}$. Then the condition $A(z, ze^{j\omega}) \neq 0$ in $|z| \leq 1$ holds if and only if the condition $A_2(z) \neq 0$ in $|z| \geq 1$ holds. The polynomial $A_2(z)$ is given by:

$$A_2(z) = z^{n+m} + c_{n+m} z^{n+m-1} + \cdots + c_3 z^2 + c_2 z + c_1$$

with

$$c_{n+m} = (\alpha_{10} + \alpha_{01} e^{j\omega})/\alpha_{00}$$

$$c_{n+m-1} = (\alpha_{20} + \alpha_{02} e^{j2\omega} + \alpha_{11} e^{j\omega})/\alpha_{00}$$

$$c_{n+m-2} = (\alpha_{30} + \alpha_{03} e^{j3\omega} + \alpha_{12} e^{j2\omega} + \alpha_{21} e^{j\omega})/\alpha_{00}$$

$$c_{n+m-3} =$$

$$(\alpha_{40}+\alpha_{04}e^{j4\omega}+\alpha_{13}e^{j3\omega}+\alpha_{31}e^{j\omega}+\alpha_{22}e^{j2\omega})/\alpha_{00}$$

$$\vdots$$

$$c_3 =$$

$$(\alpha_{n-2,m}e^{jm\omega}+\alpha_{n,m-2}e^{j(m-2)\omega}+\alpha_{n-1,m-1}e^{j(m-1)\omega})/\alpha_{00}$$

$$c_2 = (\alpha_{n,m-1}e^{j(m-1)\omega}+\alpha_{n-1,m}e^{jm\omega})/\alpha_{00}$$

$$c_1 = \alpha_{nm}e^{jm\omega}/\alpha_{00}$$

and the coefficients $c_1, c_2, \cdots, c_{n+m}$ are functions of the complex variable $e^{jl\omega}, l = 1, 2, \cdots, m$. The application of the well known Triangle Inequality with complex numbers yields

$$|(\alpha_{10}+\alpha_{01}e^{j\omega})|^2 \leq (|\alpha_{10}|+|\alpha_{01}|)^2$$

$$|(\alpha_{20}+\alpha_{02}e^{j2\omega}+\alpha_{11}e^{j\omega})|^2 \leq (|\alpha_{20}|+|\alpha_{02}|+|\alpha_{11}|)^2$$

$$|(\alpha_{30}+\alpha_{03}e^{j3\omega}+\alpha_{12}e^{j2\omega}+\alpha_{21}e^{j\omega})|^2 \leq$$

$$(|\alpha_{30}|+|\alpha_{03}|+|\alpha_{12}|+|\alpha_{21}|)^2$$

$$(|(\alpha_{40}+\alpha_{04}e^{j4\omega}+\alpha_{13}e^{j3\omega}+\alpha_{31}e^{j\omega}+\alpha_{22}e^{j2\omega})|)^2 \leq$$

$$(|\alpha_{40}|+|\alpha_{04}|+|\alpha_{13}|+|\alpha_{31}|+|\alpha_{22}|)^2$$

$$\vdots$$

$$|(\alpha_{n-2,m}e^{jm\omega}+\alpha_{n,m-2}e^{j(m-2)\omega}+\alpha_{n-1,m-1}e^{j(m-1)\omega})|^2 \leq$$

$$(|\alpha_{n-2,m}|+|\alpha_{n,m-2}|+|\alpha_{n-1,m-1}|)^2$$

$$|(\alpha_{n,m-1}e^{j(m-1)\omega}+\alpha_{n-1,m}e^{jm\omega})|^2 \leq (|\alpha_{n,m-1}|+|\alpha_{n-1,m}|)^2$$

$$|\alpha_{nm}e^{jm\omega}|^2 = |\alpha_{nm}|^2$$

We apply the sufficient condition in Lemma 1 on the polynomial $A_2(z)$ and the conditions (6) - (7) are derived.

Simple relations between the coefficients of the 2-D discrete polynomial have been presented in the proposed criterion. Other sufficient conditions for the stability of multidimensional discrete systems have been presented in the literature. The sufficient criterion proposed in this paper offers a computational efficient alternative to the existing conditions in [30], [31]. Many examples of 2-D discrete polynomials have been studied and in all cases the proposed criterion determined the stability sufficiency and rapidly as the conditions in [30], [31]. The stability checking of the 2-D discrete systems using the proposed sufficient criterion is illustrated in the following examples.

IV. EXAMPLES

Example 1

Consider the 2-D discrete system with order $(1, 2)$ and the characteristic polynomial is given by

$$A(z_1, z_2) = 2z_1 + z_2 + z_1z_2 + 3z_2^2 + 2z_1z_2^2 + 10$$

with

$\alpha_{01} = 1, \alpha_{10} = 2, \alpha_{02} = 3, \alpha_{11} = 1, \alpha_{12} = 2, \alpha_{00} = 10$

The inequality

$$(|\alpha_{01}|+|\alpha_{10}|)^2 + (|\alpha_{02}|+|\alpha_{11}|)^2 + (|\alpha_{12}|)^2 < \frac{|\alpha_{00}|^2}{3}$$

is satisfied, hence the 2-D system is BIBO stable.

Example 2

Consider the 2-D discrete system with order $(2, 2)$ and the characteristic polynomial is given by

$$A(z_1, z_2) = 2z_1+z_2+z_1^2+z_1z_2+z_2^2+z_1^2z_2+2z_1z_2^2+3z_1^2z_2^2+13$$

with

$\alpha_{01} = 1, \alpha_{10} = 2, \alpha_{02} = 1, \alpha_{11} = 1, \alpha_{20} = 1,$
$\alpha_{12} = 2, \alpha_{21} = 1, \alpha_{22} = 3, \alpha_{00} = 13$

The inequality

$$(|\alpha_{01}|+|\alpha_{10}|)^2 + (|\alpha_{02}|+|\alpha_{11}|+|\alpha_{20}|)^2 +$$

$$+(|\alpha_{12}|+|\alpha_{21}|)^2 + (|\alpha_{22}|)^2 < \frac{|\alpha_{00}|^2}{4}$$

is satisfied, hence the 2-D system is BIBO stable.

V. CONCLUSION

A new sufficient stability criterion for linear shift invariant 2-D discrete systems has been presented. This criterion offers a computational efficient alternative to the existing sufficient conditions. Using the proposed sufficient criterion, the stability checking of the 2-D discrete systems can be carried out rapidly. This has been illustrated by two detailed examples.

Acknowledgment
This work is partially funded by Ajman University.

REFERENCES

[1] A. Madanayake, C. Wijenayake, D. G. Dansereau, T. K. Gunaratne, L. T. Bruton, and S. B. Williams, “Multidimensional (MD) circuits and systems for emerging applications including cognitive radio, radio astronomy, robot vision and imaging,” *IEEE Circuits and Systems Magazine*, vol. 13, no. 1, pp. 10—43, 2013.
[2] W. Paszke, K. Galkowski, and E. Rogers, “Repetitive process based iterative learning control design using frequency domain analysis,” *Proc. IEEE International Conference on Control Applications*, pp. 1479—1484, 2012.
[3] C. U. S. Edussooriya, D. G. Dansereau, L. T. Bruton, and P. Agathoklis, “Five-dimensional depth-velocity filtering for enhancing moving objects in light field videos,” *IEEE Transactions on Signal Processing*, vol. 63, no. 8, pp. 2151-–2163, 2015.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2021.3078076, IEEE Access

[4] T. Kaczorek, *Two-dimensional linear systems* (Lecture Notes in Control and Information Sciences), vol. 68. Berlin, Germany: Springer-Verlag, 1985.
[5] W. -S. Lu, A. Antoniou, *Two-dimensional digital filters*, New York, NY, USA: Marcel Dekker, 1992.
[6] E. Fornasini, "A 2-D systems approach to river pollution modeling," *Multidimensional Systems Signal Processing*, vol. 2, no. 3, pp. 233-–265, 1991.
[7] C. Ren and S. He, "Finite-time stabilization for positive Markovian jumping neural networks," *Applied Mathematics and Computation*, vol. 365, 124631, 2020.
[8] S. He, W. Lyu and F. Liu, "Robust H_∞ sliding mode controller design of a class of time-delayed discrete conic-type nonlinear systems," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 2, pp. 885–892, 2021.
[9] C. E. De Souza, L. Xie, D. Coutinho, "Robust filtering for 2-D discrete-time linear systems with convex-bounded parameter uncertainty," *Automatica*, vol. 46, no. 4, pp. 673–681, 2010.
[10] B. Sumanasena, P. H. Bauer, "Realization using the Roesser model for implementations in distributed grid sensor networks," *Multidimensional Systems Signal Processing*, vol. 22, no.1-3, pp. 131–146, 2011.
[11] L. Li, W. Wang, "Fuzzy modeling and H_∞ control for general 2D nonlinear systems," *Fuzzy Sets and Systems*, vol. 207, pp. 1–26, 2012.
[12] E. I. Jury, "Stability of multidimensional systems and related problems," *Multidimensional Systems, Techniques and Applications*, S.G. Tzafestas, Ed. New York: Marcel Dekker, March 1986, pp. 89–159.
[13] T. S. Huang, "Stability of two-dimensional recursive filters," *IEEE Trans. Audio Electroacoust.*, vol. 20, pp. 158–163, 1972.
[14] R. A. De Carlo, J. Murray, and R. Saeks, "Multivariable Nyquist theory," *Int. J. Contr.*, vol. 25, pp. 657-675, 1977.
[15] M. G. Strintzis, "Tests of stability of multidimensional filters," *IEEE Trans. Circuits. Syst.*, vol. 24, pp. 432–437, 1977.
[16] A. Kanellakis, S. Tzafestas, and N. Theodorou, "Stability tests for 2-D systems using the Schwarz form and the inners determinants," *IEEE Transactions on Circuits and Systems*, vol. 38, no. 9, pp. 1071-–1077, 1991.
[17] Y. Bistritz, "Testing stability of 2-D discrete systems by a set of real 1-D stability tests," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 51, no. 7, pp. 1312-–1320, 2004.
[18] L. Li, L. Xu, and Z. Lin, "Stability and stabilisation of linear multidimensional discrete systems in the frequency domain," *International Journal of Control*, vol. 86, no. 11, pp. 1969—1989, 2013.
[19] X. Li, X. Hou, and M. Luo, "An explicit method for stability analysis of 2D systems described by transfer function," *IEEE Access*, vol. 7, pp. 147975—147982, 2019.
[20] L. Xu, M. Yamada, Z. Lin, O. Saito, and Y. Anazawa, "Further improvements on Bose's 2D stability test," *International Journal of Control, Automation, and Systems*, vol. 2, no. 3, pp. 319—332, 2004.
[21] P. Fu, J. Chen, and S.-I. Niculescu, "Generalized eigenvalue-based stability tests for 2-D linear systems: Necessary and sufficient conditions," *Automatica*, vol. 42, no. 9, pp. 1569-–1576, 2006.
[22] N. E. Mastorakis, I. F. Gonos and M. N. S. Swamy, "Stability of multidimensional systems using Genetic Algorithms," *IEEE Trans. Circuits. Syst.-I: Fundamental Theory and Applications*, vol.50, no.7, pp.962–965, 2003.
[23] B. Dumitrescu, "Stability test of multidimensional discrete-time systems via sum-of-squares decomposition," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol.53, no.4, pp. 928–936, 2006.
[24] Y. Bouzidi, A. Quadrat and F. Roullier, "Computer algebra methods for testing the structural stability of multidimensional systems," *IEEE 9th International Workshop on Multidimensional (nD) Systems*, pp. 1–6, 2015.
[25] Y. Bouzidi, A. Quadrat, and F. Rouillier, "Certified non-conservative tests for the structural stability of discrete multidimensional systems," *Multidimensional Systems Signal Processing*, vol. 30, no. 3, pp. 1205—1235, 2019.
[26] I. Serban and M. Najim, "A new multidimensional Schur-Cohn type stability criterion," *Proceedings of the 2007 American Control Conference*, pp. 5534–5538, New York City, USA, 2007.
[27] E. Rodriguez-Angeles, "On Stability of Multivariate Polynomials," *Systems, Structure and Control*, Ed. Petr Husek, pp. 191–206, 2008.
[28] J. A Torres-Munoz, E. Rodriguez-Angeles, and V. L. Kharitonov, "On Schur Stable Multivariate Polynomials," *IEEE Transactions on Circuits and Systems-I*, vol.53, no.5, pp.1166–1173, 2006.
[29] F. J. Kraus and P. Agathoklis, "Stability Testing of 2D Filters based on Tschebyscheff polynomials and Generalized Eigenvalues," *Proc. IEEE Int. Symp. On Circuits And Systems*, Sevilla, Spain, 2020.
[30] H. C. Reddy, P. K. Rajan, and M. N. S. Swamy, "A Simple Sufficient Criterion for the Stability of Multidimensional Digital Filters," *Proceedings of the IEEE*, vol. 70, no. 3, pp. 301–302, 1982.
[31] D. Hertz and E. Zeheb, "Sufficient Conditions for Stability of Multidimensional Discrete Systems," *Proceedings of the IEEE*, vol. 72, no. 2, p. 226, 1984.
[32] J. J. Murray, "Spectral factorization and quarter-plane digital filters," *IEEE Trans. Circuits. Syst.*, vol. 25, pp. 5867–592, 1978.
[33] P. Delsarte, Y. V. Genin and Y. Kamp, "Two variable stability criteria," Proc. IEEE Int. Symp. On Circuits And Systems, pp. 495–498, Tokyo, 1979.
[34] C. S. Berger, "Proof of a certain conjecture on the stability of linear discrete systems," *Int. J. Control*, vol. 36, pp. 545–546, 1982.
[35] A. J. Kanellakis and N. J. Theodorou, "Simple sufficient test for stability of 2-dimensional recursive digital filters," *IEE Proceedings, Pt.-G*, vol. 134, no. 5, pp. 246–247, 1987.
[36] D. Goodman, "Some stability properties of two-dimensional linear shift invariant digital filters," *IEEE Transactions on Circuits and Systems*, vol. 24, pp. 201–208, 1977.
[37] S. R.Garcia, J. Mashreghi and W. T. Ross, Finite Blaschke products and their connections, Springer, 2018.

APOSTOLOS KANELLAKIS received the diploma degree in Electrical Engineering and the PhD degree from the National Technical University of Athens, Greece in 1986 and 1992, respectively. From 1993 until 1994, he was with the University of Victoria, Canada as a Post-Doctoral Fellow. For his post-doctoral research, he received a Fellowship from the Natural Science and Engineering Research of Canada NSERC. He worked as consultant engineer in the industry from 1994 to 2019. He is a researcher at School of Electrical and Computer Engineering, National Technical University of Athens, Greece since 2019. His research interests are in multidimensional systems, stability and robustness analysis of dynamical systems, model reduction and control systems. Dr. Kanellakis is a member of the Technical Champer of Greece.

AYMAN TAWFIK is the chair of the Electrical and Computer Engineering Department, College of Engineering and Information Technology, Ajman University, UAE. He has PhD in Electrical Engineering from University of Victoria, Canada, 1995. He has MSc and BSc from Ain Shams University, Egypt, 1989 and 1983 respectively. He has over 30-year teaching experience and vast experience in Electrical Engineering program accreditation. He published more than fifty research papers in renowned journals and conferences. He worked as a consultant for DND in Canada and Egetronic company in Egypt. His research interests are DSP, digital image processing, VLSI signal processing, IoT, computer organization and education technology.

• • •