Variational Iteration Method for Solving the Space- and Time-Fractional KdV Equation

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This paper presents numerical solutions for the space- and time-fractional Korteweg–de Vries equation (KdV for short) using the variational iteration method. The space- and time-fractional derivatives are described in the Caputo sense. In this method, general Lagrange multipliers are introduced to construct correction functionals for the problems. The multipliers in the functionals can be identified optimally via variational theory. The iteration method, which produces the solutions in terms of convergent series with easily computable components, requiring no linearization or small perturbation. The numerical results show that the approach is easy to implement and accurate when applied to space- and time-fractional KdV equations. The method introduces a promising tool for solving many space–time fractional partial differential equations. Numer Methods Partial Differential Eq 24: 262–271, 2008

Keywords: KdV equation; variational iteration method; Lagrange multiplier; fractional differential equations; Caputo fractional derivative

I. INTRODUCTION

Consider the general Korteweg–de Veries (KdV) equation of the form

$$u_t + (p+1)(p+2)u^p u_x + u_{xxx} = g(x,t), \tag{1.1}$$

where $g(x,t)$ is a given function and $p = 1, 2, \ldots$ with u, u_x, $u_{xx} \to 0$ as $|x| \to \infty$. If $p = 0$, $p = 1$, and $p = 2$, Equation (1.1) becomes linearized KdV, nonlinear KdV, and modified KdV equation, respectively. The nonlinear KdV equation has been the focus of considerable recent studies for finding exact solution in Refs. [1–3] as well as numerical solution in Refs. [4–6].

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Ordinary and partial differential equations of fractional order have been the focus of many studies because of their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics, and engineering. Fractional calculus in mathematics is a natural extension of integer-order calculus and gives a useful mathematical tool for modeling many processes in nature. One of these processes, in which fractional derivatives have been successfully applied, is called diffusion [7]. The nature of the diffusion is characterized by the nonlinear dependence of the mean-square displacement $x(t)$ of a diffusing particle over time t: $\langle x^2(t)\rangle \sim t^\alpha$ for $0 < \alpha \leq 2$. This is the opposite of classical diffusion where the linear dependence $\langle x^2(t)\rangle \sim t$ occurs. Analyzing changes in the parameter α it may be said that the transport phenomena in systems exhibiting subdiffusion have $0 < \alpha < 1$, and $1 < \alpha < 2$ in the systems exhibiting superdiffusion.

The objective of the present paper is to extend the application of the variational iteration method to provide approximate solutions for the nonlinear KdV equation with time- and space-fractional derivatives of the form [8]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon u \frac{\partial^\beta u}{\partial x^\beta} + \nu \frac{\partial^3 u}{\partial x^3} = g(x,t), \quad t > 0, \; 0 < \alpha, \; \beta \leq 1, \tag{1.2}$$

where ε and ν are parameters and α and β are parameters describing the order of the fractional time and space derivatives, respectively. The function $u(x,t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t < 0$ and $x < 0$. The fractional derivatives are considered in the Caputo sense. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses. In the case of $\alpha = 1$ and $\beta = 1$, the fractional equation reduces to the classical nonlinear KdV equation.

The variational iteration method was first proposed by He [9–15], and was successfully applied to autonomous ordinary differential equations in Ref. [14], to nonlinear wave equations [15], to Helmholtz equation [16], to generalized Burger–Fisher and Burger equations [17], and other fields. Recently, the application of the variational iteration method is successfully extended to obtain analytical approximate solutions to linear and nonlinear differential equations of fractional order [18–20]. A comparison between the variational iteration method and Adomian decomposition method for solving fractional differential equations is given in Refs. [19,20]. The fact that the variational iteration method solves nonlinear equations without using Adomian polynomials can be considered as an advantage of this method over Adomian decomposition method.

The structure of this paper is as follows. We begin by introducing some basic definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results. In Section 3 we extend the application of the variational iteration method to construct our numerical solutions for the fractional nonlinear KdV equation. In Section 4 we present three examples to show the efficiency and simplicity of the method.

II. BASIC DEFINITIONS

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, \ x > 0,$$
$$J^0 f(x) = f(x).$$

Properties of the operator J^α can be found in Refs. [21–24], we mention only the following. For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by Caputo in his work on the theory of viscoelasticity [25].

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{2.1}$$

for $m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0, f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 2.1. If $m-1 < \alpha \leq m, m \in N$ and $f \in C_\mu^m$, $\mu \geq -1$, then

$$D_*^\alpha J^\alpha f(x) = f(x),$$

and,

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

The Caputo fractional derivatives are considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider the one-dimensional time- and space-fractional nonlinear KdV equation (1.2), where the unknown function $u = u(x,t)$ is a assumed to be a causal function of time and space, and the fractional derivatives are taken in Caputo sense as follows.

Definition 2.4. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_{*t}^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, & \text{for } m-1 < \alpha < m \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in \mathbb{N} \end{cases} \tag{2.2}$$

and the space-fractional derivative operator of order $\beta > 0$ is defined as

$$D_{*x}^{\beta}u(x,t) = \frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}} = \begin{cases} \frac{1}{\Gamma(m-\beta)}\int_0^x (x-\theta)^{m-\beta-1}\frac{\partial^m u(\theta,t)}{\partial\theta^m}d\theta, & \text{for } m-1<\beta<m \\ \frac{\partial^m u(x,t)}{\partial x^m}, & \text{for } \beta = m \in \mathbb{N} \end{cases} \tag{2.3}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

III. ANALYSIS OF THE VARIATIONAL ITERATION METHOD

The principles of the variational iteration method and its applicability for various kinds of differential equations are given in Refs. [9–20]. To solve the fractional KdV equation by means of variational iteration method, rewrite equation (1.2) in the form

$$(u(x,t))_{\alpha t} + \varepsilon u(x,t)(u(x,t))_{\beta x} + \nu(u(x,t))_{xxx} = g(x,t), \quad t>0, \ x>0, \tag{3.1}$$

with the initial condition

$$u(x,0) = f(x), \tag{3.2}$$

where $0 < \alpha, \beta \leq 1$. The correction functional for Eq. (3.1) can be approximately expressed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial}{\partial\tau}u_n(x,\tau) + \varepsilon\tilde{u}_n(x,\tau)(\tilde{u}_n(x,\tau))_{\beta x} + \nu(\tilde{u}_n(x,\tau))_{xxx} - g(x,\tau)\right)d\tau, \tag{3.3}$$

where λ is a general Lagrange multiplier [26], which can be identified optimally via variational theory [26–29], here $\tilde{u}_n(x,\tau)$, $(\tilde{u}_n(x,\tau))_{\beta x}$ and $(\tilde{u}_n(x,\tau))_{xxx}$ are considered as restricted variations. Making the above functional stationary,

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(\xi)\left(\frac{\partial}{\partial\tau}u_n(x,\tau) - g(x,\tau)\right)d\tau, \tag{3.4}$$

yields the following Lagrange multipliers

$$\lambda = -1.$$

Therefore, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}u_n(x,\tau) + \varepsilon u_n(x,\tau)(u_n(x,\tau))_{\beta x} + \nu(u_n(x,\tau))_{xxx} - g(x,\tau)\right)d\tau, \tag{3.5}$$

If we start with the initial approximations $u_0(x,t) = f(x)$, then the approximations $u_n(x,t)$, for $n \geq 1$, can be completely determined. Finally, we approximate the solution $u(x,t) = lim_{n\to\infty}u_n(x,t)$ by the Nth term $u_N(x,t)$.

The method provides the solution in the form of a rapidly convergent series that may lead to the exact solution in the case of linear differential equations and to an efficient numerical solution with high accuracy for nonlinear equations. The series solution begins with a trial function with possible unknown parameters. Wazwaz [30] and Abassy et al. [31] suggested some modifications to improve convergence speed for the series solutions obtained.

IV. NUMERICAL EXPERIMENTS

To incorporate our discussion above, three special cases of the fractional KdV equation (1.2) will be studied. All the results are calculated by using the symbolic calculus software Mathematica.

Example 4.1. Consider the following time-fractional KdV equation

$$D_{*t}^{\alpha}u + 6uu_x + u_{xxx} = 0, \quad 0 < \alpha \leq 1, \quad t > 0, \tag{4.1}$$

subject to the the initial condition

$$u(x,0) = \tfrac{1}{2}\text{sech}^2\left(\tfrac{1}{2}x\right). \tag{4.2}$$

The exact solution, for the special case $\alpha = 1$, is given by

$$u(x,t) = \tfrac{1}{2}\text{sech}^2\left(\tfrac{1}{2}(x-t)\right). \tag{4.3}$$

According to Eq. (3.5), the iteration formula for Eq. (4.1) is given by

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(D_t^{\alpha}u_n(x,\tau) + 6u_n(x,\tau)(u_n(x,\tau))_x + (u_n(x,\tau))_{xxx}\right)d\tau. \tag{4.4}$$

By the above variational iteration formula, begin with $u(x,0) = \frac{1}{2}\text{sech}^2(\frac{1}{2}x)$, we can obtain the following approximations

$$\begin{aligned}
u_0(x,t) &= \frac{1}{2}\text{sech}^2\left(\frac{1}{2}x\right),\\
u_1(x,t) &= \frac{1}{2}\text{sech}^2\left(\frac{1}{2}x\right) + \frac{1}{2}\text{sech}^2\left(\frac{1}{2}x\right)\tanh\left(\frac{1}{2}x\right)t,\\
u_2(x,t) &= \frac{1}{2}\text{sech}^2\left(\frac{1}{2}x\right) + \text{sech}^2\left(\frac{1}{2}x\right)\tanh\left(\frac{1}{2}x\right)t - \frac{1}{2}\text{sech}^2\left(\frac{1}{2}x\right)\tanh\left(\frac{1}{2}x\right)\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\\
&\quad + \left(\frac{1}{4}\text{sech}^2\left(\frac{1}{2}x\right) - \frac{3}{8}\text{sech}^4\left(\frac{1}{2}x\right)\right)t^2\\
&\quad + \left(\frac{1}{2}\text{sech}^4\left(\frac{1}{2}x\right) - \frac{3}{4}\text{sech}^6\left(\frac{1}{2}x\right)\right)\tanh\left(\frac{1}{2}x\right)t^3,\\
&\ \ \vdots
\end{aligned}$$

and so on, in the same manner the rest of components of the iteration formula (4.4) can be obtained using the Mathematica package.

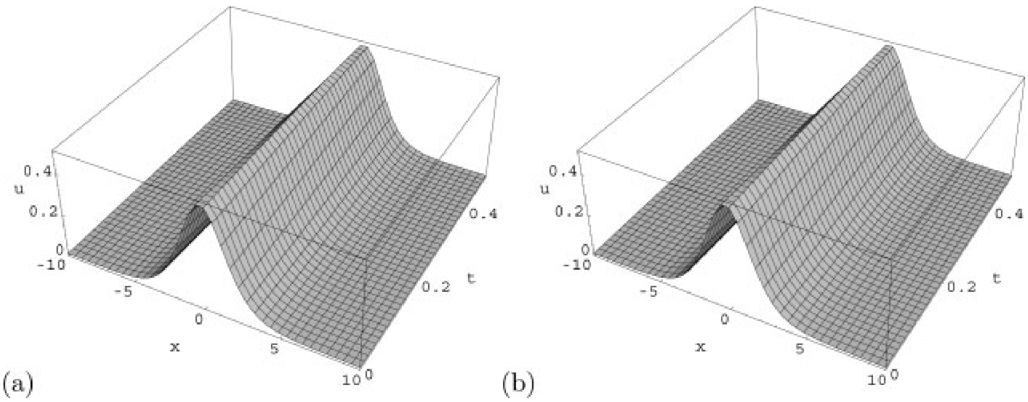

FIG. 1. The surface shows the solution $u(x, t)$ for Eq. (4.1) when $\alpha = 1$: (a) exact solution and (b) approximate solution.

The evolution results for the exact solution (4.3) and the approximate solution obtained using the variational iteration method, for the special case $\alpha = 1$, are shown in Fig. 1. It can be seen from Fig. 1 that the solution obtained by the present method is nearly identical with the exact solution. Figure 2(a, b) show the approximate solutions when $\alpha = 0.5$ and $\alpha = 0.35$, respectively. It is to be noted that only the third-order term of the variational iteration solution was used in evaluating the approximate solutions for Figs. 1 and 2. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of $u(x, t)$ when the variational iteration method is used.

Example 4.2. In this example we consider the following space- and time-fractional KdV equation

$$D_{*t}^{\alpha} u + u D_{*x}^{\beta} u + u_{xxx} = 0, \quad 0 < \alpha, \beta \leq 1, \;\; t > 0, \;\; x \geq 0, \tag{4.5}$$

subject to the the initial condition

$$u(x, 0) = x^2. \tag{4.6}$$

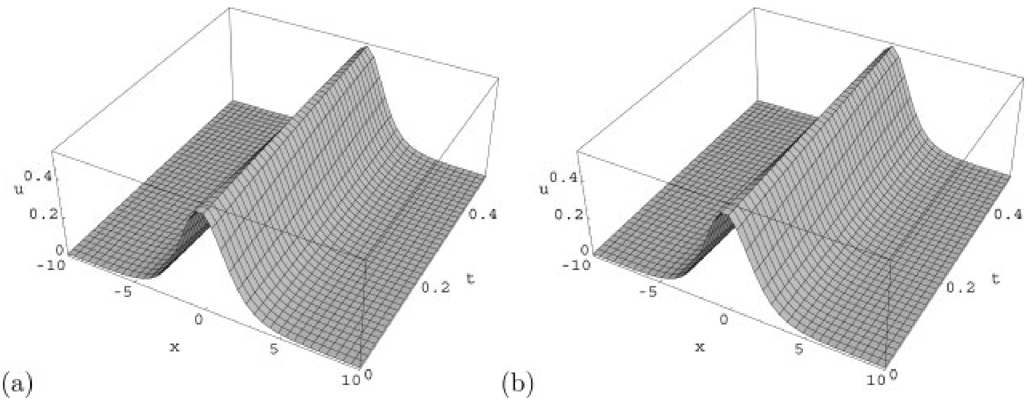

FIG. 2. The surface shows the solution $u(x, t)$ for Eq. (4.1): (a) $\alpha = 0.5$ and (b) $\alpha = 0.35$.

According to Eq. (3.5), the iteration formula for Eq. (4.5) is given by

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(D_{*t}^{\alpha}(x,\tau) + u(x,\tau)D_x^{\beta}u(x,\tau) + u_{xxx}(x,\tau)\right)d\tau. \tag{4.7}$$

By the above variational iteration formula, begin with $u(x,0) = x^2$, we can obtain the following approximations

$$\begin{aligned} u_0(x,t) &= x^2, \\ u_1(x,t) &= x^2 - 2\frac{x^{4-\beta}}{\Gamma(3-\beta)}t, \\ u_2(x,t) &= x^2 - 4\frac{x^{4-\beta}}{\Gamma(3-\beta)}t + \left(\left[\frac{\Gamma(5-\beta)}{\Gamma(5-2\beta)} + \frac{2}{\Gamma(3-\beta)}\right]\frac{x^{6-2\beta}}{\Gamma(3-\beta)} + \frac{(3-\beta)(4-\beta)x^{1-\beta}}{\Gamma(2-\beta)}\right)t^2 \\ &\quad - \frac{4}{3}\frac{\Gamma(5-\beta)x^{8-3\beta}}{\Gamma(3-\beta)^2\Gamma(5-2\beta)}t^3 + 2\frac{x^{4-\beta}}{\Gamma(3-\beta)}\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}, \\ &\quad \vdots \end{aligned}$$

and so on, in the same manner the rest of components of the iteration formula (4.7) can be obtained using the Mathematica package.

In particular, when $\alpha = 1$, the above solution reduces to the solution obtained in [8] by using the decomposition method. Figure 3(a,b) show the approximate solutions for Eq. (4.5) obtained for different values of α, and β using the variational iteration method. Comparison of Fig. 3(a,b) shows that the solution continuously depends on the fractional derivatives.

Example 4.3. We next consider the linear inhomogeneous fractional KdV equation

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} + \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 2t\ \cos(x), \quad t > 0, \quad x \in R,\ 0 < \alpha \le 1, \tag{4.8}$$

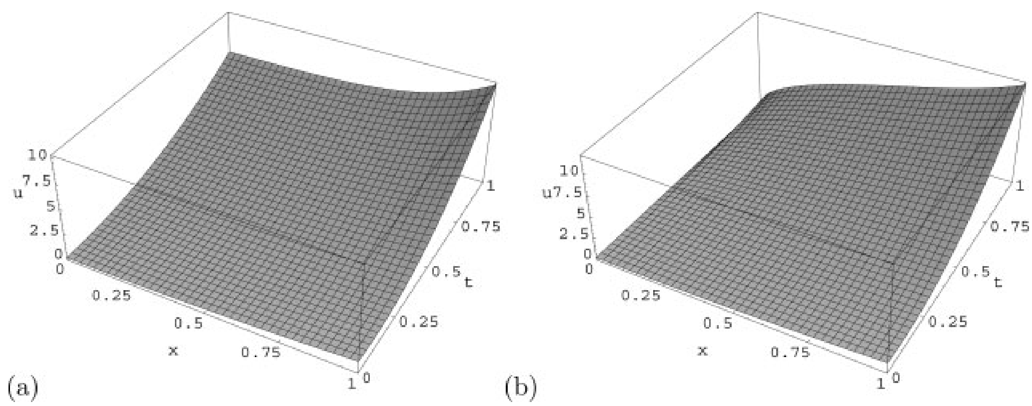

FIG. 3. The surface shows the solution $u(x,t)$ for Eq. (4.5): (a) $\beta = \alpha = 1$ and (b) $\beta = 0.5$, $\alpha = 0.75$.

subject to the initial condition

$$u(x,0) = 0. \tag{4.9}$$

According to Eq. (3.5), the iteration formula for Eq. (4.8) is given by

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(D_t^\alpha u(x,\tau) + u_x(x,\tau) + u_{xxx}(x,\tau) - \frac{2\tau^{2-\alpha}}{\Gamma(3-\alpha)} \cos(x) \right) d\tau. \tag{4.10}$$

By the above variational iteration formula, begin with $u(x,0) = 0$, we can obtain the following approximations

$$\begin{aligned} u_0(x,t) &= 0, \\ u_1(x,t) &= 2\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} \cos(x), \\ u_2(x,t) &= \left(4\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} - 2\frac{t^{4-2\alpha}}{\Gamma(5-2\alpha)} \right) \cos(x), \\ u_3(x,t) &= \left(6\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} - 6\frac{t^{4-2\alpha}}{\Gamma(5-2\alpha)} + 2\frac{t^{5-3\alpha}}{\Gamma(6-3\alpha)} \right) \cos(x), \\ &\vdots \end{aligned}$$

It is easily observed that the self-canceling "noise" terms appear between various components when $\alpha = 1$. Canceling the noise terms and keeping the nonnoise terms yields the exact solution of (4.8), for this special case, given by

$$u(x,t) = t^2 \cos(x), \tag{4.11}$$

which is easily verified.

Figure 4(a,b) show the evolution results for $\alpha = 1$ and $\alpha = 0.5$, respectively.

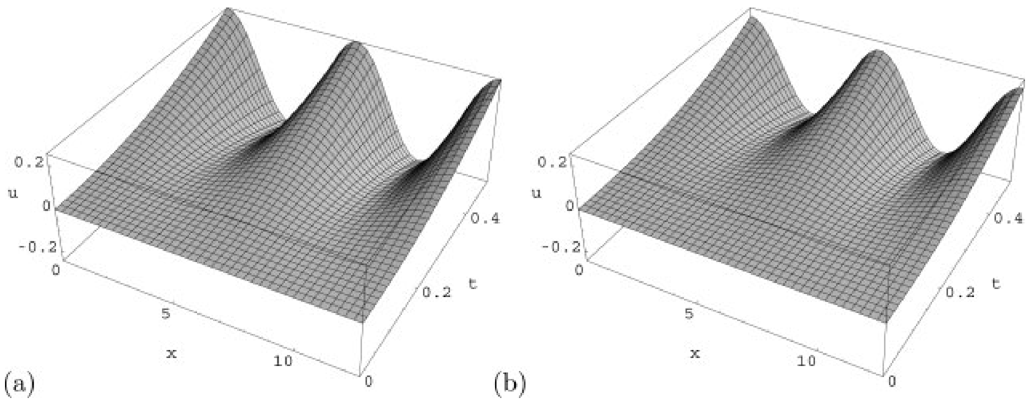

FIG. 4. The surface shows the solution $u(x,t)$ for Eq. (4.8): (a) $\alpha = 1$ and (b) $\alpha = 0.5$.

V. CONCLUSIONS

The variational iteration method was employed successfully for solving the time- and space-fractional KdV equation. The work emphasized our belief that the method is a reliable technique to handle linear and nonlinear fractional differential equations. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. The results of this method are in good agreement with those obtained by using the Adomian decomposition method. As an advantage of this method over the Adomian decomposition method, in this method we do not need to do the difficult computation for finding the Adomian polynomials. Generally speaking, the proposed method is promising and applicable to a broad class of linear and nonlinear problems in the theory of fractional calculus.

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