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Using an enhanced homotopy perturbation method in fractional differential equations via deforming the linear part

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ABSTRACT

Convergence and stability are main issues when an asymptotical method like the Homotopy Perturbation Method (HPM) has been used to solve differential equations. In this paper, convergence of the solution of fractional differential equations is maintained. Meanwhile, an effective method is suggested to select the linear part in the HPM to keep the inherent stability of fractional equations. Riccati fractional differential equations as a case study are then solved, using the Enhanced Homotopy Perturbation Method (EHPM). Current results are compared with those derived from the established Adams–Bashforth–Moulton method, in order to verify the accuracy of the EHPM. It is shown that there is excellent agreement between the two sets of results. This finding confirms that the EHPM is powerful and efficient tool for solving nonlinear fractional differential equations.

1. Introduction

Fractional differential equations are increasingly used to model problems in fluid flow, visco-elasticity, finance, engineering, and other areas of applications [1–7]. This kind of problem is more complex than ordinary differential equations. Such applications can be stated in earthquake oscillation [1], Riccati [2,3], wave [4], and chaos equations and in the control engineering [5].

The Homotopy Perturbation Method (HPM or He's Homotopy) is an asymptotical method of solving linear and nonlinear problems. Several problems have recently been solved using He's method [1–12]. *Momani* and *Odibat* have developed a modification form of the HPM (MHPM) [2]. Nonetheless, convergence is seriously questioned as a fundamental shortcoming of asymptotical method [3]. In [11] the convergence of the HPM by a proper selection of the linear part is guaranteed. A routine algorithm is presented to stabilize the linear part to keep the inherent convergence of the nonlinear equation, even when the usual part is doing vice versa. In this paper, the proposed Enhanced Homotopy Perturbation Method (EHPM) [11] is used to handle non-linear fractional differential equations. Furthermore, comparisons are made between the present method and the Adams–Bashforth–Moulton method, in order to verify the efficiency of the present method.

This paper is organized as follows: some basic definitions of fractional Differential Equations are introduced in Section 2. Rewritten numerical methods will be illustrated in Section 3. Section 4 is devoted to describe the proposed novel algorithm. An application is shown in Section 5, when the fractional Riccati is successfully solved by the presented algorithm. Finally, the work will be concluded at Section 6.

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2. Basic definition and preliminary

The differ-integral operator ${}_aD_t^q$, a combination of differentiation and integration operator, is commonly used in fractional calculus. This operator represents both the fractional differential equation and the fractional integral in a single expression; [13] defines it:

$$ {}_aD_t^q = \begin{cases} \dfrac{\mathrm{d}^q}{\mathrm{d}t^q} & q > 0 \\ 1 & q = 0 \\ \displaystyle\int_a^t (\mathrm{d}\tau)^{-q} & q < 0. \end{cases} \tag{1} $$

There are several definitions for fractional differential equations [13]. The three most commonly used definitions are the Grunwald–Letnikov, Riemann–Liouville, and Caputo definitions.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in [0, \infty)$. Clearly $C_\mu \subset C_\beta$ if $\beta \le \mu$.

Definition 2.2. A function $f(x)$, $x > 0$, is said to be in the space C_μ^m, $m \in N \cup \{0\}$, if $f^{(m)} \in C_\mu$.

Definition 2.3. The Grunwald–Letnikov fractional differential equation operator of order q [13]:

$$ {}_aD_t^q f(t) = \frac{\mathrm{d}^q f(t)}{\mathrm{d}(t-a)^q} = \lim_{N\to\infty} \left[\frac{t-a}{N}\right]^{-q} \sum_{j=0}^{N-1} (-1)^j f\left(t - j\left[\frac{t-a}{N}\right]\right). \tag{2} $$

Definition 2.4. The left sided Riemann–Liouville fractional differential equation of order $q \ge 0$, of a function $f \in C_q$, $q \ge -1$, is defined as [13]:

$$ {}_aD_t^q f(t) = \begin{cases} \dfrac{1}{\Gamma(-q)} \displaystyle\int_a^t (t-\tau)^{-q-1} f(\tau)\mathrm{d}\tau & q < 0 \\ f(t) & q = 0 \\ D^n \left[{}_aD_t^{q-n} f(t)\right] & q > 0 \end{cases} \tag{3} $$

where, n is the smallest integer larger than q, i.e., $n - 1 \le q < n$ and Γ is the Gamma function:

$$ \Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t. \tag{4} $$

For a wide class of functions, the Grunwald–Letnikov and the Riemann–Liouville definitions are equivalent [13].

Definition 2.5. Let $f \in C_{-1}^m$, $m \in N$. Then the (left sided) Caputo fractional differential equation of $f(x)$ is defined as [13]:

$$ {}_0D_t^q f(t) = \begin{cases} \dfrac{1}{\Gamma(m-q)} \displaystyle\int_0^t (t-\tau)^{m-q-1} \frac{\mathrm{d}^m f(\tau)}{\mathrm{d}\tau^m} \mathrm{d}\tau, & m-1 < q < m \\ \dfrac{\mathrm{d}^m u(x,t)}{\mathrm{d}t^m}, & q = m \in N \end{cases} \tag{5} $$

where, m is the smallest integer larger than q. Primarily, the Caputo fractional differential equation computes an ordinary differential equation, followed by a fractional integral to achieve the desired order of fractional derivative, and then the Riemann–Liouville fractional differential equation is computed in the reverse order. The Caputo fractional differential equation allows traditional initial and boundary conditions to be included in the formulation of the problem, but for homogeneous initial condition assumption, these two operators coincide. For more details on the geometric and physical interpretation for fractional differential equations of both the Riemann-Liouville and Caputo types, see [13].

3. Numerical method of solving the fractional differential equations [14]

Unlike the numerical procedure in ordinary differential equations, the numerical evaluation of fractional differential equations is quite complex. In [14] an approximation method is proposed to solve fractional differential equations numerically. This method is in essence an improved version of Adams–Bashforth–Moulton algorithm [15–17]. It is based on the predictor–correctors scheme [17,18]. Although the following proposed numerical procedure has been used to solve some specific problems, it will certainly be used for similar equations. As a practical experience, this method is found as a fundamental algorithm for these types of problem. The method will be explained systematically through some examples. Consider the following fractional differential equation:

$$D^{\alpha}y(t) = r(t, y(t)), \quad 0 \le t \le T, m-1 < \alpha \le m,$$
$$y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \ldots, m-1. \tag{6}$$

The solution of the above differential equation is equivalent to the Volterra integral series [19]:

$$y(t) = \sum_{k=0}^{\lceil\alpha\rceil-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} r(s, y(s))\mathrm{d}s. \tag{7}$$

The step size is equally spaced by $h = T/N$ where, $t_n = nh\,(n = 0, 1, \ldots, N)$. Then Eq. (7) can be rewritten as follows:

$$y_h(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} y_0^{(k)} \frac{t^k}{k!} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \left\{ r(t_{n+1}, y_h^p(t_{n+1})) + \sum_{j=0}^{n} a_{j,n+1} r(t_j, y_h(t_j)) \right\}, \tag{8}$$

where,

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n \\ 1, & j = n+1 \end{cases}$$
$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha}((n+1-j)^{\alpha} - (n-j)^{\alpha}), \tag{9}$$

and,

$$y_h^p(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} r[t_j, y_h(t_j)]. \tag{10}$$

The error of this approximation is of order p, which can be described [14] by following relation

$$O(h^p) = \max_{j=0,1,\ldots,N} \left| y(t_j) - y_h(t_j) \right| \tag{11}$$

where, $p = \min(2, 1+\alpha)$.

4. Enhanced homotopy perturbation method for fractional differential equations

The convergence is one of the main issues of asymptotic methods especially, in HPM. This method has been criticism by some authors; see [3] and the references therein. However, some effort has dealt with this issue [10]. In this section, we extend the application of the enhanced homotopy perturbation method to provide approximate solutions for nonlinear fractional differential equations. The stability of the method is a first goal to be maintained when it is applied to fractional derivatives.

4.1. A stability requirement of Fractional derivatives

A fractional order linear time invariant (FO-LTI) system may be defined in the following state-space format:

$$\begin{cases} D^{\alpha}x = Ax + Bu \\ y = Cx \end{cases} \tag{12}$$

where, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$ and $y \in \mathbb{R}^p$ denote states, input and output vectors of the system will be shown by $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$ and $C \in \mathbb{R}^{p\times n}$ respectively, and α is the fractional commensurate order. Fractional order differential equations are at least as stable as their integer orders counterparts, because systems with memory are typically more stable than their memory-less alternatives [20]. It has been shown that the autonomous dynamic $D^{\alpha}x = Ax, x(0) = x_0$ is asymptotically stable if the following condition is met [21]:

$$|\arg(\mathrm{eig}(A))| > \alpha\pi/2, \tag{13}$$

where, $0 < \alpha < 1$ and $\mathrm{eig}(A)$ represents the eigenvalues of matrix A. In this case, each component of states decays towards 0, like $t^{-\alpha}$. In addition, this system is stable if $|\arg(\mathrm{eig}(A))| \ge \alpha\pi/2$ and those critical eigenvalues which satisfy $|\arg(\mathrm{eig}(A))| > \alpha\pi/2$ have geometric multiplicity of 1. The stability regions for $0 < \alpha < 1$ are shown in Fig. 1.

Now, consider the following autonomous commensurate fractional order system:

$$D^{\alpha}x = f(x), \tag{14}$$

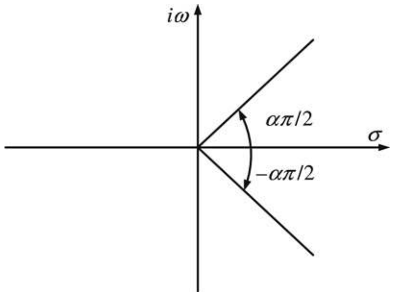

Fig. 1. Stability region of the FO-LTI system with fractional order, $0 < \alpha < 1$.

where $0 < \alpha < 1$ and $x_2 \in R^n$. The equilibrium points of system (14) can be found by solving the following equation:

$$f(x) = 0. \tag{15}$$

These points are locally asymptotically stable if all eigenvalues of the Jacobian matrix $J = \partial f/\partial x$, which are evaluated at the equilibrium points-satisfying the following condition [20,21]:

$$|\arg(\mathrm{eig}(J))| > \alpha\pi/2. \tag{16}$$

4.2. The proposed method

To describe the use of EHPM [11], consider the fractional differential equation in the following form:

$$D^{\alpha}u(t) + F(u(t), \ldots, u^{(m)}(t)) = f(t), \quad t > 0, m - 1 < \alpha < m, \tag{17}$$

where, α and m stand for rational and integer numbers, respectively. In order to use He's homotopy method, linear and nonlinear parts need to be distinguished. The method begins with a suggestion to consider the fractional term $D^{\alpha}u(t)$ as a part of the nonlinear statement. The method is continued by evaluation of the equilibrium points. As a novel contribution, the linear part will be established, using the linearized dynamic at the stable equilibrium point. This technique generally changes the right hand side of Eq. (17) into the following representation:

$$\begin{aligned} &F(u(t), \ldots, u^{(m)}(t)) = L(u(t), \ldots, u^{(m)}(t)) + N_1(u(t), \ldots, u^{(m)}(t)) \\ &\Rightarrow \underbrace{L(u(t), \ldots, u^{(m)}(t))}_{\text{Linear part}} + \underbrace{N_1(u(t), \ldots, u^{(m)}(t)) + D^{\alpha}u(t)}_{\text{Non Linear part: } N(u(t),\ldots,u^{(m)}(t))}. \end{aligned} \tag{18}$$

The homotopy statement is as the following structure:

$$H(v, p) = L(v) - L(u_0) + p[N(v) + L(u_0) - f(t)] = 0, \quad p \in [0, 1], \tag{19}$$

where, u_0 is an initial approximation and p is the small parameter varied in [0, 1]. An auxiliary variable $\nu = p^0\nu_0 + p^1\nu_1 + p^2\nu_2 + \cdots$ is substituted for the homotopy statement (19). The resultant equation is rearranged, in terms of the ascending power of p and shown the following equations:

$$\begin{aligned} &p^0 : L(v_0) - L(u_0) = 0, \quad v_0^k = A_k, \\ &p^1 : L(v_1) + N_1(v_0) + D^{\alpha}v_0 + L(u_0) - f(t) = 0, \quad v_1^k = 0, \\ &p^i : L(v_i) + N_1(v_0, \ldots, v_{i-1}) + D^{\alpha}v_{i-1} = 0, \quad v_i^k = 0, \ \forall i = 2, 3, \ldots. \end{aligned} \tag{20}$$

The appropriate solutions are then combined linearly by the following equation, to establish an asymptotic solution:

$$u(t) = \lim_{p \to 1} v(t) = v_0 + v_1 + v_2 + \cdots. \tag{21}$$

This approach will be implemented on Riccati fractional derivatives.

5. EHPM usage in the Riccati Fractional Differential Equation (RFDE)

5.1. Case study number 1

Consider the following Riccati Fractional Differential Equation (RFDE) [2,3]:

$$D^{\alpha}u + u^2 = 1, \qquad u(0) = 0, \quad 0 < \alpha < 1. \tag{22}$$

To construct the linear part, it is necessary to evaluate the Jacobian matrix at the operating point (Eq. (24)). The point is suggested to be the equilibrium point(s), which will be found by equating the dynamic (23) to zero. This is then shown as:

$$D^{\alpha}u = 1 - u^2 = 0 \rightarrow u_e = \pm 1. \tag{23}$$

The Jacobian matrix will be found at the operating point, according to the following assignment:

$$A = \left[\frac{\partial(1-u^2)}{\partial u}\right]_{u=u_e}. \tag{24}$$

To find a stable equilibrium point for the fractional equation, the argument of corresponding eigenvalues must satisfy $|\arg(\lambda)| > \alpha\pi/2$. This will be definitely achieved in Eq. (26) by:

$$A = [-2u_e]_{u_e=-1} = [2] \rightarrow \det(\lambda I - A) = 0 \rightarrow \lambda = 2 > 0 \tag{25}$$
$$|\arg(\lambda)| = 0 < \alpha\pi/2$$

$$A = [-2u_e]_{u_e=1} = [-2] \rightarrow \det(\lambda I - A) = 0 \rightarrow \lambda = -2 = 2e^{i\pi} < 0 \tag{26}$$
$$\forall 0 < \alpha \le 1, |\arg(\lambda)| = \pi > \alpha\pi/2.$$

Since the eigenvalue in Eq. (26) is negative, it is definitely treated as a stable equilibrium point. According to the stable characteristic polynomial, i.e. $\lambda + 2$, the linear and ordinary differential equation can be constructed as:

$$L(u) = \dot{u} + 2u \tag{27}$$

to have the same characteristic polynomial. Hence, the linear and nonlinear parts are correspondingly spotted as:

$$L(u) = \dot{u} + 2u, \qquad N(u) = D^{\alpha}u + u^2 - \dot{u} - 2u, \qquad f(t) = 1. \tag{28}$$

Another novel idea is to choose $u_e = u_0 = 1$ as an initial approximation. Finally, the homotopy statement can be constructed as:

$$H(v,p) = \dot{v} + 2v - 2 + p[-\dot{v} - 2v + 1 + D^{\alpha}v + v^2] = 0. \tag{29}$$

The statement is rearranged in terms of ascending power of p, which is shown by the following equations:

$$\dot{v}_0 + 2v_0 - 2 = 0, \qquad v_0(0) = 0 \tag{30}$$

$$\dot{v}_1 + 2v_1 - \dot{v}_0 - 2v_0 + v_0^2 + D^{\alpha}v_0 + 1 = 0, \quad v_1(0) = 0 \tag{31}$$

$$\vdots \qquad \vdots$$

Although the method is general, these equations are solved when α is substituted by some numbers e.g. $\alpha = 0.5$.

$$v_0(t) = 1 - e^{-2t} \tag{32}$$

$$v_1(t) = 2te^{-2t} + 0.5e^{-4t} + i\sqrt{2}te^{-2t}\mathrm{erf}(i\sqrt{2t}) + 0.56\sqrt{t}e^{-.106e-8t} + i0.35e^{-2t}\mathrm{erf}(i\sqrt{2t}) - 0.5e^{-2t} \tag{33}$$

where, i denotes the imaginary symbol and erf(.) is the error function with the following definition:

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-\tau^2}d\tau. \tag{34}$$

A second order approximation of the HPM is yielded when two terms of auxiliary function $v(t)$ is of interest, as:

$$u(t) = v_0(t) + v_1(t) = 1 + (2t - 1.5)e^{-2t} + 0.5e^{-4t} + i\sqrt{2}te^{-2t}\mathrm{erf}(i\sqrt{2t}) + 0.56\sqrt{t}e^{-.106e-8t} + i0.35e^{-2t}\mathrm{erf}(i\sqrt{2t}). \tag{35}$$

This function is easily evaluated in Maple 10 software. In the following, a second order solution of Riccati FDE using EHPM together with the numerical method (i.e. Eq. (8)) is shown in Fig. 2, considering different values for α. Although, a second order of $v(t)$ is considered, EHPM is converged to the numerical response. The error will certainly be reduced when a higher order of $v(t)$ is of concern. The method is applied when α is varied in [0, 1]. As it can be seen in Fig. 2, though two methods of the numerical one and the EHPM maintain convergence, the transient response is conversely dependent on the variation of α. It means, the transient error at the beginning will be increased when α tends to zero. This is because for α close to zero there is a smaller degree of fractionality for the dynamic. Therefore, the system behaves less dynamically; whereas the approximation of the fraction will be made less in the 2nd term of the HPM equations (in (31)). However, the error is being rapidly reduced when the time of simulation is increased or the commensurate order i.e. α approaches one. Furthermore, the accuracy of EHPM can be increased when higher order of v is considered instead of assuming two (in (35)).

When $\alpha = 1$, the fractional differential equation is reduced to a normal differential equation of integer order. This changes the nonlinear part to a nonlinear polynomial of $N(u) = D^1u + u^2 - \dot{u} - 2u = u^2 - 2u$. It means the linear part imitates the dynamic by a first order. The homotopy technique finds the following 3rd order approximation for $\alpha = 1$ and the appropriate response in Fig. 3.

$$u(t) = v_0 + v_1 + v_2 = 1 - \frac{7}{4}e^{-2t} + e^{-4t} - \frac{1}{4}e^{-6t}. \tag{36}$$

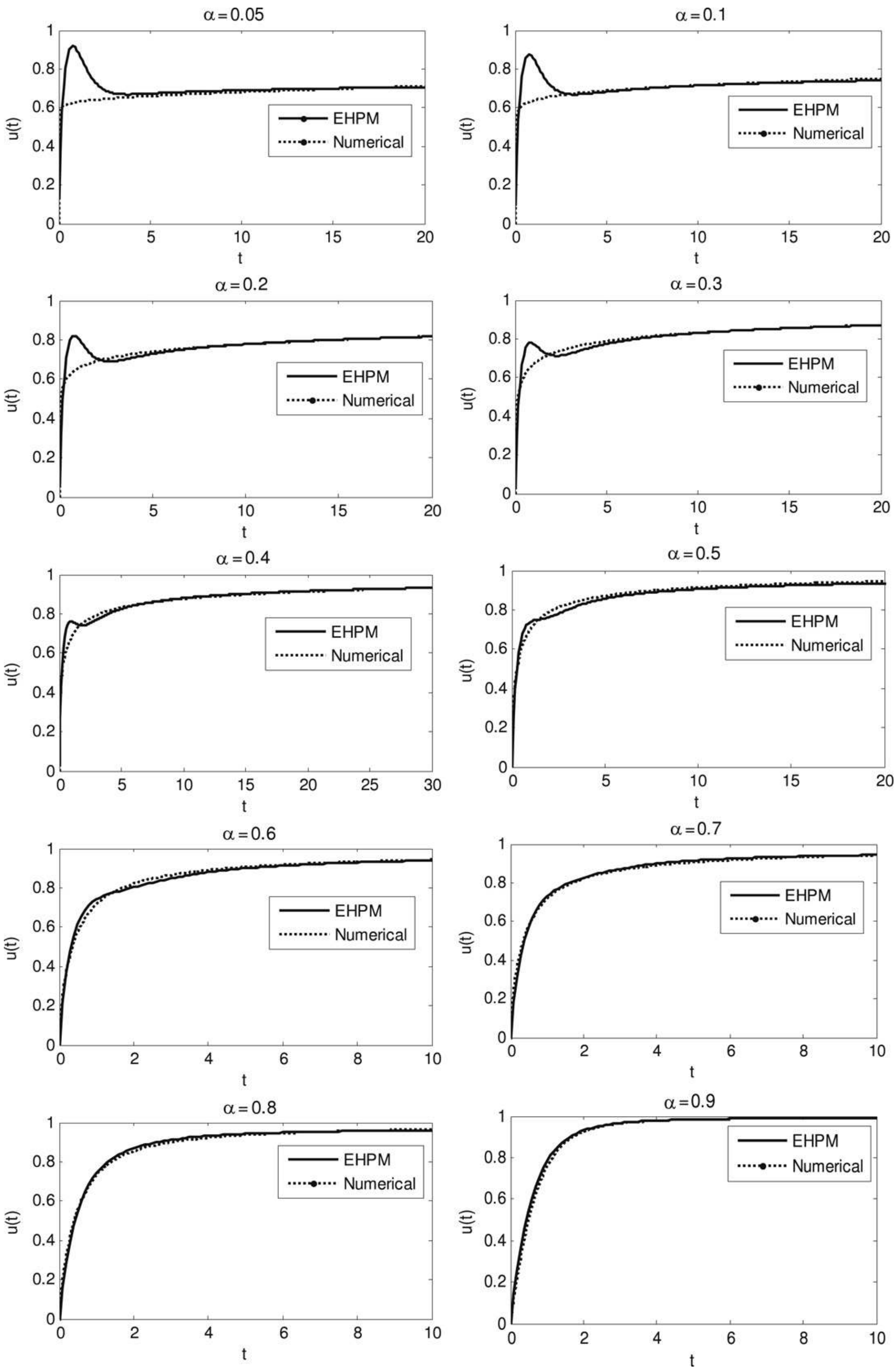

Fig. 2. Solution of the Riccati Fractional Differential equation ($D^{\alpha}u + u^2 = 1$, $u(0) = 0$) by two methods of EHPM and that of the numerical one.

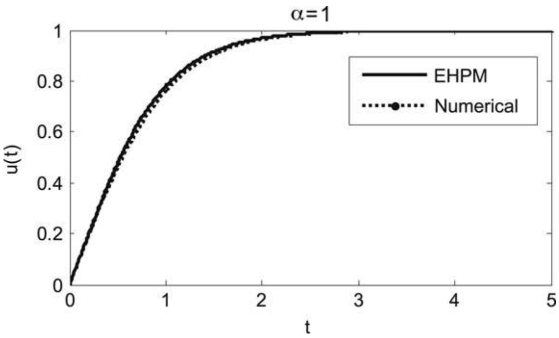

Fig. 3. Solution of the Riccati Fractional Differential equation ($\dot{u}+u^2=1$, $u(0)=0$) by two methods of EHPM and the numerical algorithm.

5.2. Second case study

Consider another Riccati Fractional Derivative, which is as follows [2,3]:

$$D^\alpha u - 2u + u^2 = 1, \qquad u(0)=0, \quad 0<\alpha<1. \tag{37}$$

The equilibrium point and the appropriate linearization will be yielded by:

$$D^\alpha u = 1+2u-u^2 = 0 \to u_e = 1 \pm \sqrt{2}. \tag{38}$$

The Jacobian matrix is found as:

$$A = \left[\frac{\partial(1+2u-u^2)}{\partial u}\right]_{u=u_e}. \tag{39}$$

The strict necessary and sufficient stability condition is met when the sign of eigenvalue is negative. In the meantime, the stable equilibrium point will be found by the following procedure:

$$A = [2-2u_e]_{u_e=1+\sqrt{2}} = [-2\sqrt{2}] \to \det(\lambda I - A) = 0 \to \lambda = -2.828 = 2.82\mathrm{e}^{\mathrm{i}\pi} \tag{40}$$
$$\forall 0<\alpha\le 1, \quad |\arg(\lambda)| = \pi > \alpha\pi/2$$

$$A = [2-2u_e]_{u_e=1-\sqrt{2}} = [2\sqrt{2}] \to \det(\lambda I - A) = 0 \to \lambda = 2.828 \tag{41}$$
$$|\arg(\lambda)| = 0 < \alpha\pi/2.$$

According to Eq. (40), the equilibrium point $u_e = 1+\sqrt{2}$ is a stable one. This is eventually because of the sign (or angle) of the corresponding eigenvalue, i.e. $\lambda = -2.828 < 0$. The relevant linear term according to the characteristic polynomial will be of the following form:

$$\Delta(\lambda) = \lambda + 2.828 \Rightarrow L(u) = \dot{u} + 2.828u. \tag{42}$$

Therefore, the linear and nonlinear parts of the homotopy statement can be recognized as:

$$L(u) = \dot{u} + 2.828u, \qquad N(u) = D^\alpha u + u^2 - \dot{u} - 4.828u, \qquad f(t) = 1. \tag{43}$$

Similarly, the initial approximation of the homotopy statement i.e. u_0 could be stated as $u_e = u_0 = 1+\sqrt{2}$. A similar procedure to that of case No. 1 yields the following Homotopy statement and the rearranged differential equations:

$$H(v,p) = \dot{v} + 2.828v - 6.82 + p[-\dot{v} - 4.828v + 5.82 + D^\alpha v + v^2] = 0. \tag{44}$$

Consequently:

$$\dot{v}_0 + 2.828v_0 - 6.82 = 0, \qquad v_0(0) = 0 \tag{45}$$

$$\dot{v}_1 + 2.828v_1 - \dot{v}_0 - 4.828v_0 + v_0^2 + D^\alpha v_0 + 5.82 = 0, \quad v_1(0) = 0 \tag{46}$$

$$\vdots \qquad\qquad \vdots$$

A solution for the fractional differential of order $\alpha = 0.5$ can be found as:

$$v_0(t) = 2.41(1-\mathrm{e}^{-2.828t}) \tag{47}$$

$$v_1(t) = \left(4.02\mathrm{i}\left(t.\mathrm{erf}(\mathrm{i}1.67\sqrt{t}) - \frac{\mathrm{i}0.337\sqrt{t}}{\mathrm{e}^{-2.828t}}\right) + 0.179\mathrm{erf}(\mathrm{i}1.67\sqrt{t})\right)$$
$$+ 0.0143\mathrm{e}^{2.828t} + 6.72t + 2.057\mathrm{e}^{-2.828t} - 2.071\mathrm{e}^{-2.828t}. \tag{48}$$

A similar 2nd order of $v(t)$ will be computed as:

$$u(t) = v_0(t) + v_1(t) = 2.41(1 - e^{-2.828t}) + \left(4.02i(t.\mathrm{erf}(i1.67\sqrt{t}) - \frac{i0.337\sqrt{t}}{e^{-2.828t}}) + 0.179\mathrm{erf}(i1.67\sqrt{t})\right) + 0.0143e^{2.828t} + 6.72t + 2.057e^{-2.828t} - 2.071e^{-2.828t}. \quad (49)$$

The EHPM and the numerical responses are shown in Fig. 4 for different α. As in the previous case, the initial transient error is increased when α approaches one. This will be compensated by considering the higher order of v in EHPM. However, convergence will be provided for all cases of variation of α. Similar to that of in the previous case, the fractional differential equation will be altered to an integer type of differential equation when $\alpha = 1$. This reduces the nonlinear part to: $N(u) = D^1u + u^2 - \dot{u} - 4.828u = u^2 - 4.828u$, and the linear part is remained untouched. A 3rd order approximation of the homotopy statement yields the following solution [22], which is shown in Fig. 5.

$$u(t) = v_0 + v_1 + v_2 = \frac{603}{250} + \left(-\frac{1911}{250} + \frac{542}{125}t - \frac{96}{125}t^2\right)e^{-2t} + \left(\frac{1086}{125} - \frac{576}{125}t\right)e^{-4t} - \frac{432}{125}e^{-6t}. \quad (50)$$

In continuing, the performance of the proposed method will be shown when it is compared with the Modified HPM (MHPM) [2].

5.3. The Comparison between EHPM and MHPM

Cases in 5.1 and 5.2 have been solved using MHPM [2]. At this method, $\dot{y}$ is only considered as a linear term. Since this term is unstable, this is insufficient. Therefore, more terms should be considered to overcome this shortcoming. The MHPM solutions of 5.1 and 5.2 are respectively shown in Eqs. (51) and (52).

$$\begin{aligned} u = {} & 4t - \frac{10}{3}t^3 + \frac{4}{5}t^5 - \frac{17}{315}t^7 - 6\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \left[\frac{6}{\Gamma(3-\alpha)} + \frac{6}{\Gamma(4-\alpha)} + \frac{4\Gamma(5-\alpha)}{\Gamma(4-\alpha)^2}\right]\frac{\Gamma(4-\alpha)}{\Gamma(5-\alpha)}t^{4-\alpha} \\ & - \left[\frac{2}{3\Gamma(3-\alpha)} + \frac{4}{\Gamma(4-\alpha)} + \frac{16}{\Gamma(6-\alpha)}\right]\frac{\Gamma(6-\alpha)}{\Gamma(7-\alpha)}t^{6-\alpha} \\ & - \left[\frac{1}{\Gamma(3-\alpha)^2} + \frac{1}{\Gamma(4-2\alpha)} + \frac{2\Gamma(5-\alpha)}{\Gamma(4-\alpha)\Gamma(5-2\alpha)}\right]\frac{\Gamma(5-2\alpha)}{\Gamma(6-2\alpha)}t^{5-2\alpha} + 4\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} - \frac{t^{4-2\alpha}}{\Gamma(5-3\alpha)} \end{aligned} \quad (51)$$

$$\begin{aligned} u = {} & 4t + 6t^2 - \frac{2}{3}t^3 - 3t^4 + \frac{t^5}{15} + \frac{34t^6}{90} - \frac{17t^7}{315} - 6\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{3-\alpha}}{\Gamma(4-\alpha)} \\ & + \left[\frac{10}{\Gamma(3-\alpha)} - \frac{2}{\Gamma(4-\alpha)}\right]\frac{\Gamma(4-\alpha)}{\Gamma(5-\alpha)}t^{4-\alpha} \\ & + \left[\frac{2}{\Gamma(3-\alpha)} + \frac{8}{\Gamma(4-\alpha)} + \frac{20}{\Gamma(5-\alpha)} + \frac{4\Gamma(4-\alpha)}{\Gamma(3-\alpha)\Gamma(5-\alpha)}\right]\frac{\Gamma(5-\alpha)}{\Gamma(6-\alpha)}t^{5-\alpha} \\ & - \left[\frac{2}{3\Gamma(3-\alpha)} + \frac{4}{\Gamma(5-\alpha)} + \frac{16}{\Gamma(6-\alpha)} + \frac{4\Gamma(4-\alpha)}{\Gamma(3-\alpha)\Gamma(5-\alpha)}\right]\frac{\Gamma(6-\alpha)}{\Gamma(7-\alpha)}t^{6-\alpha} \\ & + 4\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} + 6\frac{t^{4-2\alpha}}{\Gamma(5-2\alpha)} \\ & - \left[\frac{1}{\Gamma(3-\alpha)^2} + \frac{2}{\Gamma(4-2\alpha)} + \frac{2}{\Gamma(5-2\alpha)} + \frac{2\Gamma(4-\alpha)}{\Gamma(3-\alpha)\Gamma(5-\alpha)}\right]\frac{\Gamma(5-2\alpha)}{\Gamma(6-2\alpha)}t^{5-2\alpha} - \frac{t^{4-3\alpha}}{\Gamma(5-3\alpha)} \end{aligned} \quad (52)$$

where $\Gamma(x)$ is well-known gamma function which defines as:

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\mathrm{d}t. \quad (53)$$

A comparison between the EHPM vs. the MHPM [2] for the two cases is shown in Tables 1 and 2 whilst the numerical is treated as the actual dynamic. These are plotted in Fig. 6. From the numerical results in Fig. 6, it is to be noted that the EHPM solution follows the numerical one whereas MHPM needs more modification. Two main points may be spotted; the convergence of the solution and negligible error. In the same method, when α approaches unity, the approximation becomes more accurate. However, the proposed method shows some more improvement over other methods. The performance shall be promoted if a similar [2] 4th order approximation is used. Besides, the proposed technique is simpler. These examples signify the quality of the new proposed method.

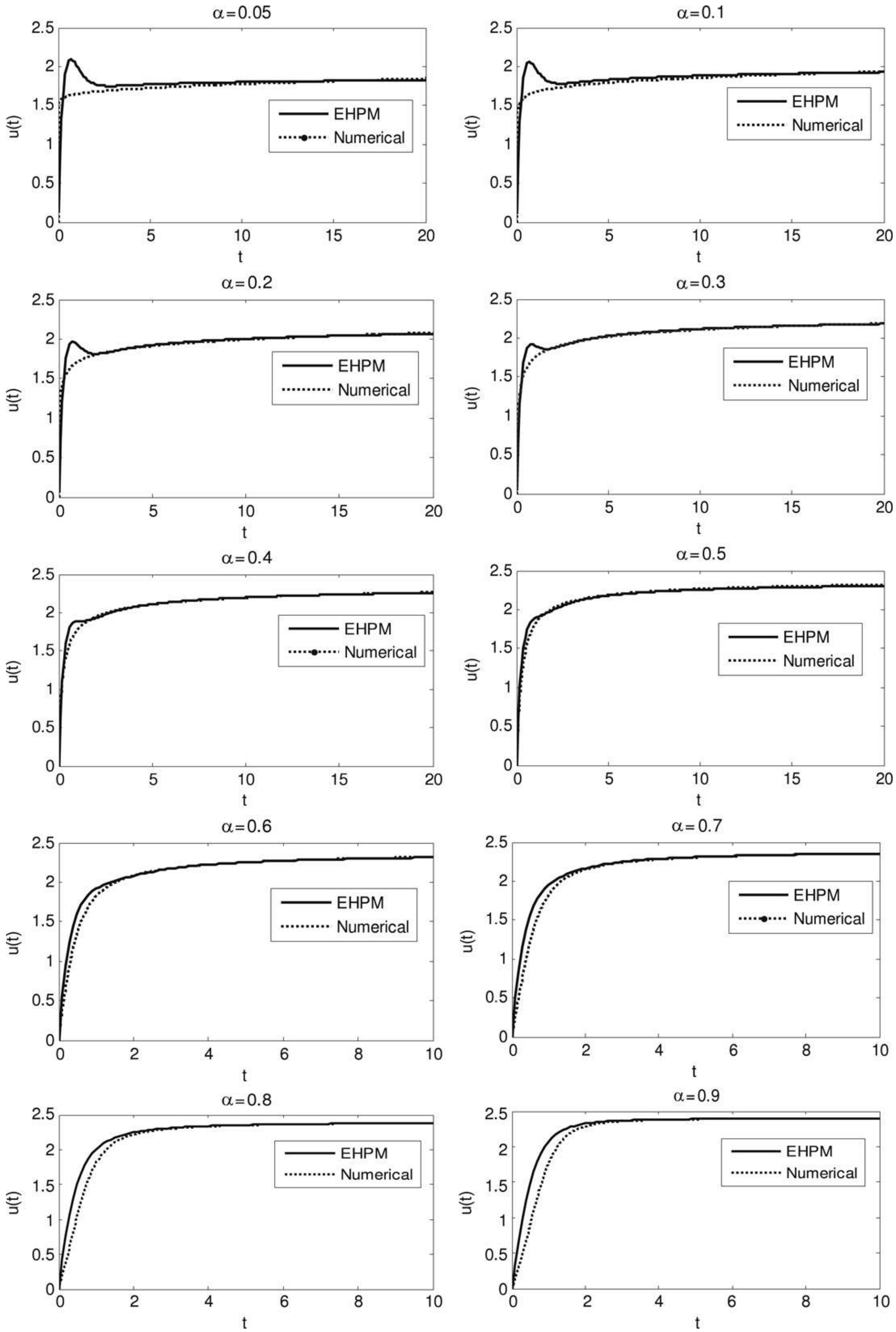

Fig. 4. Solution of the Riccati Fractional Differential equation ($D^{\alpha}u - 2u + u^2 = 1, u(0) = 0$) by two methods of EHPM and the numerical algorithm.

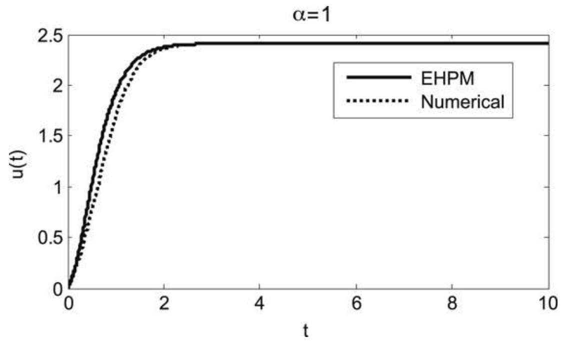

Fig. 5. Solution of the Riccati Fractional Differential equation ($\dot{u} - 2u + u^2 = 1$, $u(0) = 0$) by two methods of EHPM and the numerical algorithm.

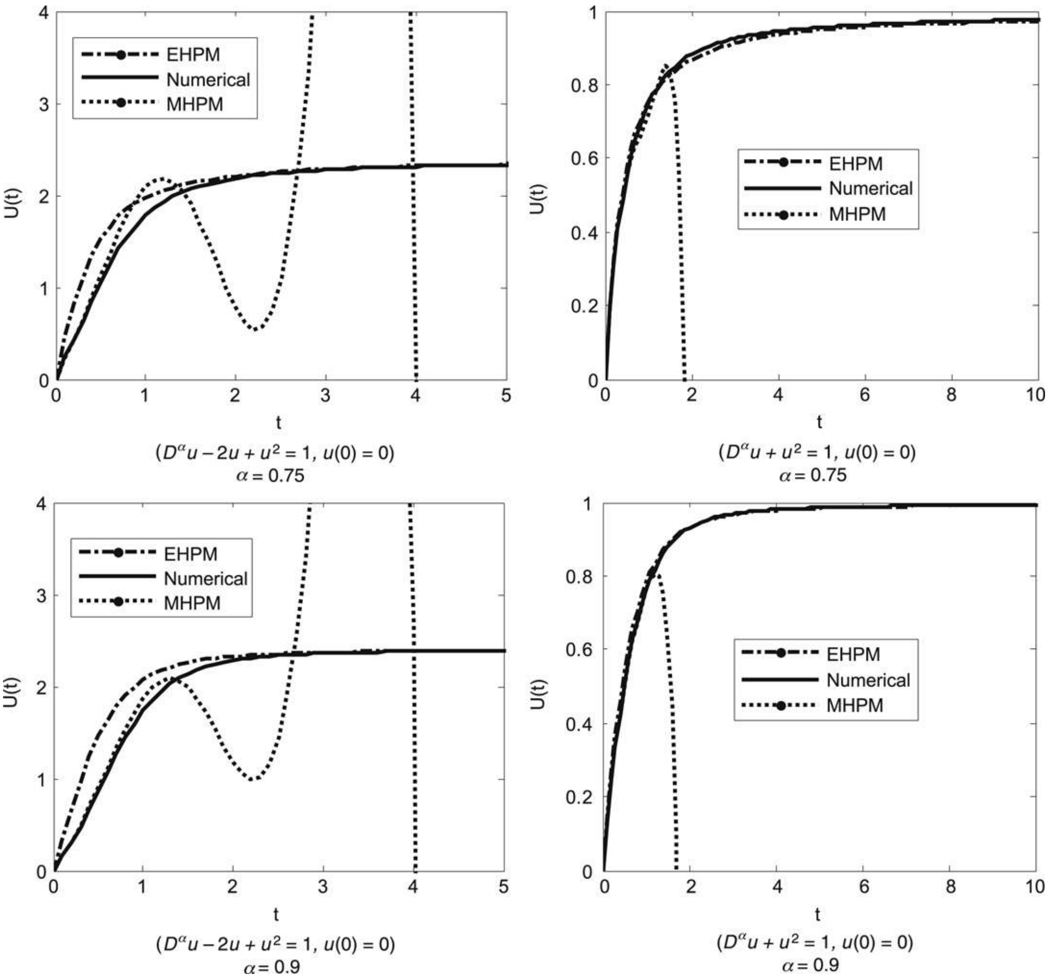

Fig. 6. Graphs of values in Tables 1 and 2, considering 3 methods of the numerical, MHPM and EHPM.

6. Conclusion

He's homotopy perturbation is enhanced and used to solve fractional differential equations. The enhanced algorithm has successfully been implemented to find approximate solutions for Riccati fractional equations. The work emphasized our belief that the proposed method is a reliable technique for handling nonlinear differential equations of fractional order. Finally, the recent appearance of fractional differential equations as models in some fields of engineering [1] makes it necessary to investigate the method for solutions of such equations (analytical and numerical) and we hope that this work is a step in this direction.

Table 1
Solution of Riccati fractional differential equation, case No. 1 that is achieved by EHPM and MHPM [2] together with the numerical one

$D^\alpha u + u^2 = 1, u(0) = 0$						
t	$\alpha = 0.75$			$\alpha = 0.9$		
	Numerical	EHPM	MHPM	Numerical	EHPM	MHPM
0	0	0	0	0	0	0
0.2	0.3117	0.3214	0.3138	0.2393	0.2647	0.2391
0.4	0.4855	0.5077	0.4929	0.4234	0.4591	0.4229
0.6	0.6045	0.6259	0.5974	0.5679	0.6031	0.5653
0.8	0.6880	0.7028	0.6604	0.6774	0.7068	0.6740
1.0	0.7478	0.7542	0.7183	0.7584	0.7806	0.7569
1.2	0.7915	0.7901	0.7922	0.8174	0.8329	0.8041
1.4	0.8240	0.8165	0.8520	0.8600	0.8702	0.7549
1.6	0.8486	0.8369	0.7597	0.8909	0.8971	0.4462
1.8	0.8678	0.8534	0.1914	0.9134	0.9167	−0.4603
2.0	0.8828	0.8672	−1.4670	0.9299	0.9313	−2.5757
2.2	0.8949	0.8790	−5.2528	0.9421	0.9422	−6.9095
2.4	0.9048	0.8891	−12.7916	0.9513	0.9507	−15.0277
2.6	0.9130	0.8980	−26.4988	0.9584	0.9572	−29.2446
2.8	0.9198	0.9058	−49.8197	0.9638	0.9624	−52.8547
3.0	0.9256	0.9126	−87.5126	0.9681	0.9666	−90.4055
3.2	0.9306	0.9186	−145.9774	0.9715	0.9700	−148.0163
3.4	0.9349	0.9239	−233.6347	0.9743	0.9728	−233.7461
3.6	0.9386	0.9286	−361.3562	0.9766	0.9751	−358.0135
3.8	0.9420	0.9328	−542.9534	0.9785	0.9770	−534.0727
4.0	0.9449	0.9364	−795.7247	0.9801	0.9787	−778.5492

Table 2
Solution of Riccati fractional differential equation, case No. 2 that is achieved by EHPM and MHPM [2] together with the numerical one

$D^\alpha u - 2u + u^2 = 1, u(0) = 0$						
t	$\alpha = 0.75$			$\alpha = 0.9$		
	Numerical	EHPM	MHPM	Numerical	EHPM	MHPM
0	0	0	0	0	0	0
0.5000	1.0622	1.5209	1.1328	0.8621	1.4614	0.9010
1.0000	1.7780	1.9753	2.0874	1.7356	2.0697	1.8720
1.5000	2.0631	2.1297	1.9017	2.1424	2.2587	1.9844
2.0000	2.1818	2.2062	0.7787	2.2848	2.3251	1.1817
2.5000	2.2407	2.2520	1.0917	2.3374	2.3535	1.4451
3.0000	2.2747	2.2813	5.8102	2.3604	2.3681	5.6630
3.5000	2.2967	2.3012	12.3698	2.3725	2.3767	12.1453
4.0000	2.3121	2.3153	−0.0807	2.3798	2.3824	2.6812
4.5000	2.3236	2.3258	−92.1386	2.3848	2.3865	−77.9674
5.0000	2.3324	2.3340	−396.4145	2.3884	2.3896	−353.2532
5.5000	2.3395	2.3406	-1.1627×10^3	2.3912	2.3920	-1.0588×10^3
6.0000	2.3453	2.3460	-2.8182×10^3	2.3934	2.3940	-2.6014×10^3
6.5000	2.3502	2.3505	-0.6045×10^4	2.3952	2.3956	-0.5635×10^4
7.0000	2.3543	2.3544	-1.1876×10^4	2.3967	2.3969	-1.1155×10^4
7.5000	2.3578	2.3578	-2.1813×10^4	2.3979	2.3981	-2.0617×10^4
8.0000	2.3609	2.3608	-3.7971×10^4	2.3990	2.3991	-3.6072×10^4
8.5000	2.3637	2.3634	-0.6324×10^5	2.3999	2.4000	-0.6034×10^5
9.0000	2.3661	2.3657	-1.0148×10^5	2.4008	2.4008	-0.9719×10^5
9.5000	2.3683	2.3678	-1.5776×10^5	2.4015	2.4015	-1.5157×10^5
10.0000	2.3702	2.3697	-2.3858×10^5	2.4022	2.4021	-2.2988×10^5

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