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The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics

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ABSTRACT

Variational iteration method has been used to handle linear and nonlinear differential equations. The main property of the method lies in its flexibility and ability to solve nonlinear equations accurately and conveniently. In this work, a general framework of the variational iteration method is presented for analytical treatment of fractional partial differential equations in fluid mechanics. The fractional derivatives are described in the Caputo sense. Numerical illustrations that include the fractional wave equation, fractional Burgers equation, fractional KdV equation, fractional Klein–Gordon equation and fractional Boussinesq-like equation are investigated to show the pertinent features of the technique. Comparison of the results obtained by the variational iteration method with those obtained by Adomian decomposition method reveals that the first method is very effective and convenient. The basic idea described in this paper is expected to be further employed to solve other similar linear and nonlinear problems in fractional calculus.

1. Introduction

Recent advances of fractional differential equations are stimulated by new examples of applications in fluid mechanics, viscoelasticity, mathematical biology, electrochemistry and physics. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [1], and the fluid-dynamic traffic model with fractional derivatives [2] can eliminate the deficiency arising from the assumption of continuum traffic flow. Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested in Ref. [3], and differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena [4]. Different fractional partial differential equations have been studied and solved including the space–time fractional diffusion-wave equation [5–7], the fractional advection-dispersion equation [8,9], the fractional telegraph equation [10], the fractional KdV equation [11] and the linear inhomogeneous fractional partial differential equations [12].

The Adomian decomposition method [13–17] and the variational iteration method [18–38] are relatively new approaches to provide an analytical approximation to linear and nonlinear problems, and they are particularly valuable as tools for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of analytical solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization. The decomposition method has been used to obtain approximate solutions of a large class of linear or nonlinear differential equations [13,14]. Recently, the application of the method is extended for fractional differential equations [10,11,39–44]. The variational iteration method, which proposed by Ji-Huan He [19–28], was successfully applied to autonomous ordinary and partial differential equations and other fields. Ji-Huan He [3] was the first to apply the

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variational iteration method to fractional differential equations. Recently Odibat and Momani [41–47] implemented the variational iteration method to solve linear and nonlinear differential equations of fractional order.

The objective of this paper is to extend the application of the variational iteration method to obtain analytical solutions to some fractional partial differential equations in fluid mechanics. These equations include wave equation, Burgers equation, KdV equation, Klein–Gordon equation and Boussinesq-like equation. The variational iteration method is a computational method that yields analytical solutions and has certain advantages over standard numerical methods. It is free from rounding off errors as it does not involve discretization, and does not require large computer obtained memory or power. The method introduces the solution in the form of a convergent fractional series with elegantly computable terms. The corresponding solutions of the integer order equations are found to follow as special cases of those of fractional order equations.

Throughout this paper, fractional partial differential equations are obtained from the corresponding integer order equations by replacing the first-order or the second-order time derivative by a fractional in the Caputo sense [48] of order α with $0 < \alpha \leq 1$ or $1 < \alpha \leq 2$.

2. Preliminaries and notations

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(> \mu)$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \quad \alpha > 0, t > 0,$$
$$J^0 f(t) = f(t).$$

Properties of the operator J^α can be found in [49–51], we mention only the following: For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,
2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,
3. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by M. Caputo in his work on the theory of viscoelasticity [48].

Definition 2.3. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(t) \mathrm{d}t, \tag{2.1}$$

for $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, $t > 0$, $f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 2.1. *If $m - 1 < \alpha \leq m$, $m \in N$ and $f \in C_\mu^m$, $\mu \geq -1$, then*

$$D^\alpha J^\alpha f(t) = f(t),$$

and

$$J^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem [52]. In this paper, we consider the one-dimensional linear inhomogeneous fractional partial differential equations in fluid mechanics, where the unknown function $u(x, t)$ is assumed to be a causal function of time, i.e., vanishing for $t < 0$. The fractional derivative is taken in Caputo sense as follows:

Definition 2.4. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} \mathrm{d}\tau, & \text{for } m-1 < \alpha < m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in \mathbb{N}. \end{cases} \tag{2.2}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

3. Variational iteration method

The principles of the variational iteration method and its applicability for various kinds of differential equations are given in [18–38]. In [3], Ji-Huan He showed that the variational iteration method is also valid for fractional differential equations. He applied the method to obtain analytical solution for the fractional differential equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = f(x,t), \qquad u(a) = b, \quad 1 < \alpha < 2. \tag{3.1}$$

In this section, following the discussion presented in [3], we extend the application of the variational iteration method to solve the time fractional differential equation:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = R[x]u(x,t) + q(x,t), \quad t > 0, x \in \mathbf{R}, \tag{3.2}$$

where $R[x]$ is a differential operator in x, subject to the initial and boundary conditions

$$\begin{aligned} &u(x,0) = f(x), \quad 0 < \alpha \le 1, \\ &u(x,t) \to 0 \quad \text{as } |x| \to \infty, t > 0, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x), \quad 1 < \alpha \le 2, \\ &u(x,t) \to 0 \quad \text{as } |x| \to \infty, t > 0, \end{aligned} \tag{3.4}$$

where $f(x)$, $g(x)$, and $q(x,t)$ all are continuous functions and α, $m-1 < \alpha \le m$, is a parameter describing the order of the time-fractional derivative in the Caputo sense. According to the variational iteration method, we can construct the correction functional for Eq. (3.2) as:

$$\begin{aligned} u_{k+1}(x,t) &= u_k(x,t) + J_t^\beta\left[\lambda\left(\frac{\partial^\alpha}{\partial t^\alpha} u_k(x,t) - R[x]\tilde{u}_k(x,t) - q(x,t)\right)\right], \\ &= u_k(x,t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \lambda(\tau) \left(\frac{\partial^\alpha}{\partial t^\alpha} u_k(x,\tau) - R[x]\tilde{u}_k(x,\tau) - q(x,\tau)\right) \mathrm{d}\tau, \end{aligned} \tag{3.5}$$

where J_t^β is the Riemann–Liouville fractional integral operator of order $\beta = \alpha - \mathit{floor}(\alpha)$, that is $\beta = \alpha + 1 - m$, with respect to the variable t and λ is a general Lagrange multiplier, which can be identified optimally via variational theory [29]. To identify approximately Lagrange multiplier, some approximation must be made. The correction functional (3.5) can be approximately expressed as follows

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \left[\lambda(\tau)\left(\frac{\partial^m}{\partial t^m} u_k(x,\tau) - R[x]\tilde{u}_k(x,\tau) - q(x,\tau)\right)\right] \mathrm{d}\tau. \tag{3.6}$$

Here we apply restricted variations to the nonlinear term $R[x]u$, in this case we can easily determine the multiplier. Making the above functional stationary, noticing that $\delta\tilde{u}_k = 0$,

$$\delta u_{k+1}(x,t) = \delta u_k(x,t) + \delta \int_0^t \lambda(\tau)\left(\frac{\partial^m}{\partial \tau^m} u_k(x,\tau) - q(x,\tau)\right) \mathrm{d}\tau, \tag{3.7}$$

yields the following Lagrange multipliers

$$\lambda = -1, \quad \text{for } m = 1, \tag{3.8}$$

$$\lambda = \tau - t, \quad \text{for } m = 2. \tag{3.9}$$

Therefore, for $m = 1$ $(0 < \alpha \leq 1)$, we substitute $\lambda = -1$ into the functional (3.5) to obtain the following iteration formula:

$$u_{k+1}(x,t) = u_k(x,t) - J_t^{\alpha}\Big[\frac{\partial^{\alpha}}{\partial t^{\alpha}}u_k(x,t) - R[x]u_k(x,t) - q(x,t)\Big]. \tag{3.10}$$

For $m = 2$ $(1 < \alpha \leq 2)$, we substitute $\lambda = \tau - t$ into the functional (3.5) to get

$$\begin{aligned} u_{k+1}(x,t) &= u_k(x,t) + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2}(\tau-t)\Big(\frac{\partial^{\alpha}}{\partial t^{\alpha}}u_k(x,\tau) - R[x]u_k(x,\tau) - q(x,\tau)\Big)\mathrm{d}\tau, \\ &= u_k(x,t) - \frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Big(\frac{\partial^{\alpha}}{\partial t^{\alpha}}u_k(x,\tau) - R[x]u_k(x,\tau) - q(x,\tau)\Big)\mathrm{d}\tau. \end{aligned} \tag{3.11}$$

So, we obtain the following iteration formula:

$$u_{k+1}(x,t) = u_k(x,t) - (\alpha-1)J_t^{\alpha}\Big[\frac{\partial^{\alpha}}{\partial t^{\alpha}}u_k(x,t) - R[x]u_k(x,t) - q(x,t)\Big]. \tag{3.12}$$

The initial approximation (trial function) u_0 can be freely chosen if it satisfies the initial and boundary conditions of the problem. However the success of the method depends on the proper selection of the initial approximation u_0. Finally, we approximate the solution $u(x,t) = \lim_{k\to\infty} u_k(x,t)$ by the Nth term $u_N(x,t)$.

4. Decomposition method

The principles of the decomposition method and its applicability for various kinds of differential equations are given in [13–17] and the references cited therein. In this section we implement the decomposition method to solve Eq. (3.2). The decomposition method requires that the nonlinear fractional differential Eq. (3.2) be expressed in terms of operator from as

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t) = L[x]u(x,t) + N[x]u(x,t) + q(x,t), \quad t > 0, x \in \mathbf{R}, \tag{4.1}$$

where $L[x]$ is a linear operator in x and $N[x]$ is a nonlinear operator in x. The method is based on applying the operator J_t^{α}, the inverse of the operator D_t^{α}, to both sides of Eq. (4.1) to obtain

$$u(x,t) = \sum_{k=0}^{m-1}\frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J_t^{\alpha}(L[x]u(x,t) + N[x]u(x,t) + q(x,t)). \tag{4.2}$$

The Adomian decomposition method [13,14] suggests the solution $u(x,t)$ be decomposed into the infinite series of components

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{4.3}$$

and the nonlinear function in Eq. (4.2) is decomposed as follows:

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{4.4}$$

where A_n are the so-called Adomian polynomials. Substituting the decomposition series (4.3) and (4.4) into both sides of (4.2) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \sum_{k=0}^{m-1}\frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J_t^{\alpha}\left(R[x]\sum_{n=0}^{\infty}u_n(x,t) + \sum_{n=0}^{\infty}A_n + q(x,t)\right). \tag{4.5}$$

Following the decomposition method, we introduce the recursive relation as

$$\begin{aligned} u_0(x,t) &= \sum_{k=0}^{m-1}\frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J_t^{\alpha}(q(x,t)), \\ u_{j+1}(x,t) &= J_t^{\alpha}(L[x]u_j(x,t) + A_j), \quad j \geq 0. \end{aligned} \tag{4.6}$$

The Adomian polynomial A_n can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [15]. The general form of formula for A_n Adomian polynomials is

$$A_n = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}N\left(\sum_{k=0}^{n}\lambda^k u_k\right)\right]_{\lambda=0}. \tag{4.7}$$

This formula is easy to compute by using Mathematica software or by writing a computer code to get as many polynomials as we need in the calculation of the numerical as well as explicit solutions. It is worth noting that if the zeroth component u_0 is defined then the remaining components u_j, $j \geq 1$, can be completely determined such that each term is determined by using the previous terms, and the series solution is thus entirely determined. Finally, we approximate the solution $u(x, t)$ by the truncated series

$$\phi_N(x,t) = \sum_{j=0}^{N-1} u_j(x,t) \quad \text{and} \quad \lim_{N\to\infty} \phi_N(x,t) = u(x,t). \tag{4.8}$$

However, the inclusion of boundary conditions in fractional differential equations introduces additional difficulties. The Adomian decomposition method can handle these difficulties by using the time-fractional operator D_t^α and the initial conditions only. The method provides the solution in the form of a rapidly convergent series that may lead to the exact solution in the case of linear differential equations and to an efficient numerical solution with high accuracy for nonlinear equations. The convergence of the decomposition series has been investigated by several authors [53,54].

5. Applications: Linear equations

To incorporate our discussion above, three linear fractional PDEs will be studied. The decomposition method and the variational iteration method are used to construct the exact solutions of the problems.

Example 5.1. Consider the following one-dimensional linear inhomogeneous fractional wave equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) + t\cos(x), \quad t > 0, x \in R, 0 < \alpha \leq 1, \tag{5.1}$$

subject to the initial condition

$$u(x,0) = 0. \tag{5.2}$$

Following the discussion presented in the decomposition method section, we can obtain the recurrence relation

$$\begin{aligned} u_0(x,t) &= u(x,0) + J^\alpha\left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(x) + t\cos(x)\right) = t\sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x), \\ u_{j+1}(x,t) &= -J^\alpha\left(\frac{\partial}{\partial x}u_j(x,t)\right), \quad j \geq 0. \end{aligned} \tag{5.3}$$

In view of (5.3), the first few components are derived as follows:

$$\begin{aligned} u_0(x,t) &= t\sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x), \\ u_1(x,t) &= -J^\alpha\left(\frac{\partial}{\partial x}u_0(x,t)\right) = -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x), \\ u_2(x,t) &= -J^\alpha\left(\frac{\partial}{\partial x}u_1(x,t)\right) = -\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x) - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos(x), \\ &\vdots \end{aligned}$$

and so on, in this manner the rest of components of the decomposition series can be obtained.

The solution in series form is given by

$$\begin{aligned} u(x,t) = {} & t\sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x) - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x) \\ & - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x) - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos(x) + \cdots. \end{aligned} \tag{5.4}$$

It is easily observed that the self-canceling "noise" terms appear between various components. Canceling the noise terms and keeping the non-noise terms in (5.4) yields the exact solution of (5.1) given by

$$u(x,t) = t\sin(x), \tag{5.5}$$

which is easily verified. It is worth noting that other noise terms between other components of (5.4) will be canceled, as the sixth terms, and the sum of these "noise" terms will vanish in the limit. This formally justified in [14].

According to the variational iteration method and to Eq. (3.10), the iteration formula for Eq. (5.1) is given by

$$u_{k+1}(x,t) = u_k(x,t) - J_t^{\alpha}\Big[\frac{\partial^{\alpha}}{\partial t^{\alpha}}u_k(x,t) + \frac{\partial}{\partial x}u_k(x,t) - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(x) - t\cos(x)\Big]. \tag{5.6}$$

By the above variational iteration formula, if we begin with $u_0 = 0$, we can obtain the following approximations

$$\begin{aligned}
u_1(x,t) &= t\sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x),\\
u_2(x,t) &= t\sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x) - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x),\\
u_3(x,t) &= t\sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x) - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x)\\
&\quad - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x) - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos(x),\\
&\vdots
\end{aligned}$$

Canceling the noise terms and keeping the non-noise terms yield the exact solution of Eq. (5.1). If we begin with $u_0 = t\sin(x)$ then the exact solution follows immediately by using two iterations.

Example 5.2. In this example we consider the one-dimensional linear inhomogeneous fractional Burgers equation given by

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, \quad t > 0, x \in R, 0 < \alpha \le 1, \tag{5.7}$$

subject to the initial condition

$$u(x,0) = x^2. \tag{5.8}$$

Proceeding as before we obtain the recurrence relation

$$\begin{aligned}
u_0(x,t) &= u(x,0) + J^{\alpha}\Big(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\Big),\\
u_{j+1}(x,t) &= -J^{\alpha}(L_{1x}u_j(x,t) - L_{2x}u_j(x,t)), \quad j \ge 0,
\end{aligned} \tag{5.9}$$

where $L_{1x} = \frac{\partial}{\partial x}$ and $L_{2x} = \frac{\partial^2}{\partial x^2}$, so that the first few components are

$$\begin{aligned}
u_0(x,t) &= x^2 + t^2 + \frac{t^{\alpha}}{\Gamma(\alpha+1)}(2x-2),\\
u_1(x,t) &= -J^{\alpha}(L_{1x}u_0(x,t) - L_{2x}u_0(x,t)) = -\frac{t^{\alpha}}{\Gamma(\alpha+1)}(2x-2) - 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
u_2(x,t) &= -J^{\alpha}(L_{1x}u_1(x,t) - L_{2x}u_1(x,t)) = 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
u_3(x,t) &= -J^{\alpha}(L_{1x}u_2(x,t) - L_{2x}u_2(x,t)) = 0,
\end{aligned}$$

and as a result $u_j(x,t) = 0, j \ge 3$. The exact solution is therefore given by

$$u(x,t) = x^2 + t^2. \tag{5.10}$$

According to the variational iteration method and to Eq. (3.10), the iteration formula for Eq. (5.7) is given by

$$u_{k+1}(x,t) = u_k(x,t) - J_t^{\alpha}\Big[\frac{\partial^{\alpha}}{\partial t^{\alpha}}u_k(x,t) + \frac{\partial}{\partial x}u_k(x,t) - \frac{\partial^2}{\partial x^2}u_k(x,t) - \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - 2x + 2\Big]. \tag{5.11}$$

By the above variational iteration formula, if we begin with $u_0 = x^2$, we can obtain the following approximations

$$\begin{aligned}
u_1(x,t) &= x^2 + t^2,\\
&\vdots\\
u_n(x,t) &= x^2 + t^2.
\end{aligned}$$

The exact solution $u(x,t) = x^2 + t^2$ follows immediately. The success of obtaining the exact solution by using two iterations is a result of the proper selection of initial guess u_0.

Example 5.3. We consider the one-dimensional linear inhomogeneous fractional Klein–Gordon equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u = 6x^3 t + (x^3 - 6x)t^3, \quad t > 0, x \in R, 1 < \alpha \le 2, \tag{5.12}$$

subject to the initial conditions

$$u(x, 0) = 0, \qquad u_t(x, 0) = 0. \tag{5.13}$$

Following the discussion presented above, we obtain the recurrence relation

$$\begin{aligned} &u_0(x, t) = u(x, 0) + tu_t(x, 0) + J^\alpha(6x^3 t + (x^3 - 6x)t^3), \\ &u_{j+1}(x, t) = J^\alpha(L_{2x} u_j(x, t) - u_j(x, t)), \quad j \ge 0. \end{aligned} \tag{5.14}$$

In view of (5.14), the first few components are derived as follows:

$$\begin{aligned} u_0(x, t) &= 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x)\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}, \\ u_1(x, t) &= J^\alpha(L_{2x} u_0(x, t) - u_0(x, t)) = 36x\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 36x\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} \\ &\quad - 6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - (x^3 - 6x)\frac{6t^{2\alpha+3}}{\Gamma(2\alpha+4)}, \\ &\vdots \end{aligned}$$

and so on, in this manner the rest of components of the decomposition series can be obtained. Substituting the above components into (4.3), we obtain the solution in a series form

$$\begin{aligned} u(x, t) &= 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x)\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)} + 36x\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 36x\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} \\ &\quad - 6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - (x^3 - 6x)\frac{6t^{2\alpha+3}}{\Gamma(2\alpha+4)} + \cdots. \end{aligned} \tag{5.15}$$

According to the variational iteration method and to Eq. (3.12), the iteration formula for Eq. (5.12) is given by

$$u_{k+1}(x, t) = u_k(x, t) - (\alpha - 1)J_t^\alpha\left[\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u - 6x^3 t - (x^3 - 6x)t^3\right]. \tag{5.16}$$

By the above variational iteration formula, if we begin with $u_0 = 0$, we can obtain the following approximations

$$\begin{aligned} u_1(x, t) &= (\alpha - 1)\left[6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x)\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}\right], \\ u_2(x, t) &= 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 6(x^3 - 6x)\frac{t^{\alpha+3}}{\Gamma(\alpha+4)} \\ &\quad - (\alpha - 1)^2\left[6(x^3 - 6x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 6(x^3 - 12x)\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)}\right] + \cdots. \end{aligned} \tag{5.17}$$

From (5.15) and (5.17), the decomposition method and the variational iteration method give the same solution for the classical Klein–Gordon Eq. (5.12) (when $\alpha = 2$) which is given by

$$u(x, t) = x^3 t^3 + (x^3 - 6x)\frac{6t^5}{\Gamma(6)} + 36x\frac{t^5}{\Gamma(6)} - 36x\frac{6t^7}{\Gamma(8)} - 6x^3 \frac{t^5}{\Gamma(6)} - (x^3 - 6x)\frac{6t^7}{\Gamma(8)} + \cdots. \tag{5.18}$$

Canceling the noise terms and keeping the non-noise terms in (5.18) yield the exact solution of (5.12), for the special case $\alpha = 2$,

$$u(x, t) = x^3 t^3, \tag{5.19}$$

which is easily verified.

6. Applications: Nonlinear equations

For nonlinear equations in general, there exists no method that yields the exact solution and therefore only approximate solutions can be derived. In this section, we use the variational iteration method and the decomposition method to provide approximate solutions for two kinds of nonlinear time-fractional partial differential equations.

Example 6.4. We consider the time-fractional KdV equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad t > 0, x \in R, 0 < \alpha \le 1, \tag{6.1}$$

subject to the initial condition

$$u(x,0) = \frac{1}{2}\operatorname{sech}^2\left(\frac{1}{2}x\right). \tag{6.2}$$

The exact solution, for the special case $\alpha = 1$, is given by

$$u(x,t) = \frac{1}{2}\operatorname{sech}^2\left(\frac{1}{2}(x-t)\right). \tag{6.3}$$

The time-fractional KdV Eq. (6.1) is solved in [11] using the decomposition method. The solution in series form is found as

$$u(x,t) = f_0(x) + f_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{6.4}$$

where

$$\begin{aligned} f_0(x) &= \frac{1}{2}\operatorname{sech}^2\left(\frac{1}{2}x\right), \\ f_1(x) &= -6f_0f_0' - f_0''', \\ f_2(x) &= -6f_1f_0' - 6f_0f_1' - f_1''', \\ f_3(x) &= -6f_2f_0' - \frac{6\Gamma(2\alpha+1)f_1f_1'}{\Gamma(\alpha+1)^2} - 6f_0f_2' - f_2'''. \end{aligned}$$

According to the variational iteration method and to Eq. (3.10), the iteration formula for the time-fractional KdV Eq. (6.1) is given by

$$u_{k+1}(x,t) = u_k(x,t) - J_t^\alpha\left[\frac{\partial^\alpha u_k}{\partial t^\alpha} + 6u_k\frac{\partial u_k}{\partial x} + \frac{\partial^3 u_k}{\partial x^3}\right]. \tag{6.5}$$

By the above variational iteration formula, if we begin with $u_0 = \frac{1}{2}\operatorname{sech}^2\left(\frac{1}{2}x\right)$, we can obtain the following approximations

$$\begin{aligned} u_1(x,t) &= f_0(x), \\ u_2(x,t) &= f_0(x) + f_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_3(x,t) &= f_0(x) + f_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 6f_1f_1'\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}. \\ &\vdots \end{aligned}$$

Now, the fourth-order term approximate solution for Eq. (6.1) obtained using the variational iteration method is given in Eq. (6.4), which is the same solution obtained using the decomposition method. Therefore, both methods provide the same approximate solution for the time-fractional KdV equation.

Example 6.5. In this example we consider the time-fractional Boussinesq-like equation

$$D_t^\alpha u + (u^2)_{xx} - (u^2)_{xxxx} = 0, \quad t > 0, x \in R, \tag{6.6}$$

where $1 < \alpha \le 2$, subject to the initial conditions

$$u(x,0) = \frac{4}{3}\sinh^2\left(\frac{1}{4}x\right), \qquad u_t(x,0) = -\frac{1}{3}\sinh\left(\frac{1}{2}x\right). \tag{6.7}$$

The decomposition method admits the use of the recurrence relation

$$\begin{aligned} u_0(x,t) &= \frac{4}{3}\sinh^2\Big(\frac{1}{4}x\Big) + \frac{1}{3}\sinh\Big(\frac{1}{2}x\Big)t, \\ u_{j+1}(x,t) &= -J^{\alpha}\Big((A_j)_{xx} - (A_j)_{xxxx}\Big), \quad j \geq 0 \end{aligned} \tag{6.8}$$

where A_j's are the Adomian polynomials of the nonlinearity u^2, which are given by

$$\begin{aligned} A_0(x,t) &= u_0^2, \\ A_1(x,t) &= 2u_0u_1, \\ A_2(x,t) &= 2u_0u_2 + u_1^2, \\ A_3(x,t) &= 2u_0u_3 + 2u_1u_2, \\ A_4(x,t) &= 2u_0u_4 + 2u_1u_3 + u_2^2. \end{aligned}$$

Solving (6.8) recursively, as a result we obtain the following fourth-order term approximate solution for the time-fractional Boussinesq-like Eq. (6.6)

$$\begin{aligned} u(x,t) = {} & \frac{4}{3}\sinh^2\Big(\frac{1}{4}x\Big) - \frac{1}{3}\sinh\Big(\frac{1}{2}x\Big)t + \frac{1}{2.3}\cosh\Big(\frac{1}{2}x\Big)\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ & - \frac{1}{2^2.3}\sinh\Big(\frac{1}{2}x\Big)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{2^3.3}\cosh\Big(\frac{1}{2}x\Big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{1}{2^4.3}\sinh\Big(\frac{1}{2}x\Big)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ & + \frac{1}{2^5.3}\cosh\Big(\frac{1}{2}x\Big)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{1}{2^6.3}\sinh\Big(\frac{1}{2}x\Big)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}. \end{aligned} \tag{6.9}$$

According to the variational iteration method and to Eq. (3.12), the iteration formula for the time-fractional Boussinesq-like Eq. (6.6) is given by

$$u_{k+1}(x,t) = u_k(x,t) - (\alpha-1)J_t^{\alpha}\Big[\frac{\partial^{\alpha}u_k}{\partial t^{\alpha}} + \frac{\partial^2 u}{\partial x^2}u_k^2 - \frac{\partial^4}{\partial x^4}u_k^2\Big]. \tag{6.10}$$

By the above variational iteration formula, if we begin with $u_0 = \frac{4}{3}\sinh^2\Big(\frac{1}{4}x\Big) - \frac{1}{3}\sinh\Big(\frac{1}{2}x\Big)t$, we obtain the following fourth-order term approximate solution for the time-fractional Boussinesq-like Eq. (6.6)

$$\begin{aligned} u(x,t) = {} & \frac{4}{3}\sinh^2\Big(\frac{1}{4}x\Big) - \frac{1}{3}\sinh\Big(\frac{1}{2}x\Big)t + (\alpha-1)\Big[\frac{1}{2.3}\cosh\Big(\frac{1}{2}x\Big)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{1}{2^2.3}\sinh\Big(\frac{1}{2}x\Big)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\Big] \\ & + (\alpha-1)^2\Big[\frac{1}{2^3.3}\cosh\Big(\frac{1}{2}x\Big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{1}{2^4.3}\sinh\Big(\frac{1}{2}x\Big)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\Big] \\ & + (\alpha-1)^3\Big[\frac{1}{2^5.3}\cosh\Big(\frac{1}{2}x\Big)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{1}{2^6.3}\sinh\Big(\frac{1}{2}x\Big)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\Big]. \end{aligned} \tag{6.11}$$

It is interesting to point out that for the case of $\alpha = 2$, the approximate solution

$$u(x,t) = \frac{2}{3}\Big[\cosh\Big(\frac{1}{2}x\Big)\Big(1 + \frac{1}{2^2}\frac{t^2}{2!} + \frac{1}{2^4}\frac{t^4}{4!} + \cdots\Big) - 1\Big] - \frac{2}{3}\sinh\Big(\frac{1}{2}x\Big)\Big[\frac{1}{2}t + \frac{1}{2^3}\frac{t^3}{3!} + \frac{1}{2^5}\frac{t^5}{5!} + \cdots\Big],$$

follows immediately upon replacing α by 2 in the decomposition solution (6.9) or the variational iteration solution (6.11), which converges to the exact solution of the Boussinesq-like Eq. (6.6), when $\alpha = 2$,

$$u(x,t) = \frac{4}{3}\sinh^2\Big(\frac{1}{4}(x-t)\Big). \tag{6.12}$$

7. Conclusions

Variational iteration method has been known as a powerful tool for solving many functional equations such as ordinary, partial differential equations, integral equations and so many other equations. In this article, we have presented a general framework of the variational iteration method for the analytical treatment of fractional partial differential equations in fluid mechanics. The present work shows the validity and great potential of the variational iteration method for solving linear and nonlinear fractional partial differential equations. All of the examples show that the results of the variational iteration method are in excellent agreement with those obtained by the Adomian decomposition method. The basic idea described in this paper is expected to be further employed to solve other similar nonlinear problems in fractional calculus.

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