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The homotopy analysis method for handling systems of fractional differential equations

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ABSTRACT

In this paper we present an efficient numerical algorithm for solving linear and nonlinear systems of fractional differential equations (FDEs). The homotopy analysis method is applied to construct the numerical solutions. The proposed algorithm avoids the complexity provided by other numerical approaches. The method is applied to solve three systems of FDEs. Results obtained using the scheme presented here agree well with the analytical solutions and the numerical results presented elsewhere. Moreover, an attempt has been made to address few issues like the effect of varying the auxiliary parameter $\hbar$, auxiliary function $H(t)$, and the auxiliary linear operator $\mathscr{L}$ on the order of local error and convergence of the method. Also, we show that the homotopy perturbation method and Adomain decomposition method are special cases of the homotopy analysis method.

1. Introduction

Fractional order ordinary differential equations, as generalizations of classical integer order ordinary differential equations, are increasingly used to model problems in fluid flow, mechanics, viscoelasticity, biology, physics and engineering, and other applications. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models [1–9]. Fractional differentiation and integration operators are also used for extensions of the diffusion and wave equations [10]. The solutions of fractional differential equations are much involved. In general, there exists no method that yields an exact solution for fractional differential equations. Only approximate solutions can be derived using linearization or perturbation methods. Several methods have been suggested to solve fractional differential equations. These methods include the homotopy perturbation method [11–14], Adomian's decomposition method [15–19], and variation iteration method [20,21,11,22,23]. However, the convergence region of the corresponding results is rather small, as shown later in this paper. Recently, Liao [24], proposed a powerful analytic method, namely the homotopy analysis method (HAM), for solving linear and nonlinear differential and integral equations. Different from perturbation techniques, the homotopy analysis method does not depend upon any small or large parameters. Besides, it logically contains other non-perturbation techniques, such as Adomian's decomposition method [15–19], homotopy perturbation method [11–14], Lyapunov's artificial small parameter method [25], and the δ-expansion method [26], as proved by Liao [24,27–35]. The HAM was successfully applied to solve many nonlinear problems such as nonlinear Riccati differential equations with fractional order [27], nonlinear Vakhnenko equation [31], the Glauert-jet

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problem [36], fractional KdV–Burgers–Kuramoto equation [37], a generalized Hirota–Satsuma coupled KdV equation [38], nonlinear heat transfer [39], and so on. In this paper we extend the applications of HAM to solve linear and nonlinear system of fractional differential equations. Besides, we note that Adomian decomposition method and homotopy perturbation method are special cases of the HAM when $\hbar = -1$. Moreover, we investigate the effect of varying the auxiliary parameter $\hbar$, auxiliary function $H(t)$, auxiliary linear operator $\mathscr{L}$ on the order of local error and convergence of the method.

The paper is organized as follows. A brief review of the fractional calculus theory is given in Section 2. In Section 3 we use the homotopy analysis method to construct our numerical solutions for systems of linear and nonlinear fractional equations. In Section 4 we present some examples to show the efficiency and simplicity of the method.

2. Basic definitions

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$. Clearly $C_\mu \subset C_\beta$ if $\beta \leqslant \mu$.

Definition 2.2. A function $f(x)$, $x > 0$, is said to be in the space $C_\mu^m, m \in N \cup \{0\}$, if $f^{(m)} \in C_\mu$.

Definition 2.3. The left sided Riemann–Liouville fractional integral operator of order $\alpha \geqslant 0$ of a function $f \in C_\mu$, $\mu \geqslant -1$, is defined as [6]

$$\begin{aligned} J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, \ x > 0, \\ J^0 f(x) &= f(x). \end{aligned} \tag{2.1}$$

Definition 2.4. Let $f \in C_{-1}^m, m \in N \cup \{0\}$ then the Caputo fractional derivative of $f(x)$ is defined as [6–8]

$$D_*^\alpha f(x) = \begin{cases} [J^{m-\alpha} f^{(m)}(x)], & m-1 < \alpha < m, \quad m \in N, \\ \frac{d^m f(x)}{dx^m}, & \alpha = m. \end{cases} \tag{2.2}$$

Hence, we have the following properties [1–9]:

$$\begin{aligned} &1.\ J^\alpha J^v f = J^{\alpha+v} f, \quad \alpha, v \geqslant 0, \ f \in C_\mu, \ \mu \geqslant -1. \\ &2.\ J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\gamma+\alpha}, \quad \alpha > 0, \quad \gamma > -1, \ x > 0. \\ &3.\ J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0, \ m-1 < \alpha \leqslant m. \end{aligned} \tag{2.3}$$

The Caputo fractional derivative [6] is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. For more information on the mathematical properties of fractional derivatives and integrals, one can consult the mentioned references.

3. Homotopy analysis method

The homotopy analysis method which provides an analytical approximate solution is applied to various nonlinear problems [24,27–39]. In this section, we extend the applications of the homotopy analysis method to the following fractional system:

$$D_*^{\alpha_i} x_i(t) = f_i(t, x_1, \ldots, x_n), \quad i = 1,2,3,\ldots,n, \ 0 \leqslant \alpha_i \leqslant 1, \tag{1}$$

subject to the initial conditions:

$$x_i(0) = a_i, \quad i = 1,2,3,\ldots,n. \tag{2}$$

3.1. Zeroth-order deformation equation

Liao [24], construct the so-called zeroth-order deformation equation:

$$(1-q)\mathscr{L}_i[\phi_i(t,q)-x_{i0}(t)] = q\hbar_i H_i(t) N_i[\phi_i(t,q)], \quad i=1,2,3,\ldots,n, \tag{3}$$

subject to the initial conditions:

$$\phi_i(0,q) = a_i, \quad i=1,2,3,\ldots,n, \tag{4}$$

where $q \in [0,1]$ is an embedding parameter, N_i are nonlinear operators, $\mathscr{L}_i$ are auxiliary linear operators satisfy $\mathscr{L}_i(0)=0$, $x_{i0}(t)$ are initial guesses satisfy the initial conditions (2), $\hbar_i \neq 0$ are auxiliary parameters, $H_i(t) \neq 0$ are auxiliary functions and $\phi_i(t,q)$ are unknown functions.

It should be emphasized that one has great freedom to choose the initial guesses, the auxiliary linear operators $\mathscr{L}_i$, the auxiliary parameters $\hbar_i$ and the auxiliary functions $H_i(t)$.

Obviously, when $q \neq 0$, since $x_{i0}(t)$ satisfy the initial conditions (2) and $\mathscr{L}_i(0)=0$, we have

$$\phi_i(t,0) = x_{i0}(t), \quad i=1,2,3,\ldots,n, \tag{5}$$

when $q = 1$, since $\hbar_i \neq 0$ and $H_i(t) \neq 0$, the zeroth-order deformation equation (3) and (4) are equivalent to (1) and (2), hence

$$\phi_i(t,1) = x_{i0}(t), \quad i=1,2,3,\ldots,n. \tag{6}$$

Thus, as q increasing from 0 to 1, the solutions $\phi_i(t,q)$ various from $x_{i0}(t)$ to $x_i(t)$. Expanding $\phi_i(t,q)$ in Taylor series with respect to the embedding parameter q, one has

$$\phi_i(t,q) = x_{i0}(t) + \sum_{m=1}^{\infty} x_{im}(t) q^m, \quad i=1,2,3,\ldots,n, \tag{7}$$

where

$$x_{im}(t) = \frac{1}{m!} \frac{\partial^m \phi_i(t,q)}{\partial q^m}\bigg|_{q=0}, \quad i=1,2,3,\ldots,n. \tag{8}$$

Assume that the auxiliary parameters $\hbar_i$, the auxiliary functions $H_i(t)$, the initial approximations $x_{i0}(t)$ and the auxiliary linear operators $\mathscr{L}_i$ are properly chosen so that the series (7) converges at $q = 1$. Then at $q = 1$, and by (6) the series (7) becomes

$$x_i(t) = x_{i0}(t) + \sum_{m=1}^{\infty} x_{im}(t), \quad i=1,2,3,\ldots,n. \tag{9}$$

3.2. The mth-order deformation equation

Define the vector

$$\vec{x}_i = \{x_{i0}(t), x_{i1}(t), x_{i2}(t), \ldots, x_{ij}(t)\}, \quad i=1,2,3,\ldots,j. \tag{10}$$

Differentiating equations (3) m times with respect to the embedding parameter q, then setting $q = 0$ and dividing them by $m!$, finally using (8), we have the so-called mth-order deformation equations:

$$\mathscr{L}_i[x_{im}(t) - \chi_m x_{i(m-1)}(t)] = \hbar_i H_i(t) R_{im}(\vec{x}_{i(m-1)}(t)), \quad i=1,2,3,\ldots,n, \tag{11}$$

subject to the initial conditions:

$$x_{im}(0) = 0, \quad i=1,2,3,\ldots,n, \tag{12}$$

where

$$R_{im}(\vec{x}_{i(m-1)}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N_i(\phi_i(t,q))}{\partial q^{m-1}}\bigg|_{q=0}, \quad i=1,2,3,\ldots,n, \tag{13}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & o.w. \end{cases} \tag{14}$$

If we choose

$$\mathscr{L}_i = D_*^{\alpha_i}, \quad i = 1, 2, \ldots, n,$$

and according to (11) we have

$$J^{\alpha_i} D_*^{\alpha_i}[x_{im}(t) - \chi_m x_{i(m-1)}(t)] = \hbar_i J^{\alpha_i}[H_i(t) R_{im}(\vec{x}_{i(m-1)}(t))], \quad i = 1, 2, 3, \ldots, n. \tag{15}$$

Using the property (2.3) and the initial conditions (12) we further have

$$x_{im}(t) = \chi_m x_{i(m-1)}(t) + \hbar_i J^{\alpha_i}[H_i(t) R_{im}(\vec{x}_{i(m-1)}(t))], \quad i = 1, 2, 3, \ldots, n, \tag{16}$$

subject to the initial conditions:

$$x_{im}(0) = 0, \quad i = 1, 2, 3, \ldots, n. \tag{17}$$

For the special case if $\alpha_i = 1$, $i = 1,2,\ldots,n$, then (16) has the form

$$x_{im}(t) = \chi_m x_{i(m-1)}(t) + \hbar_i \int_0^t [H_i(t) R_{im}(\vec{x}_{m-1}(\tau))] d\tau, \quad i = 1, 2, 3, \ldots, n.$$

4. Numerical results

In order to assess both the accuracy and the convergence order of the homotopy analysis method presented in this paper for fractional systems of differential equations, we have applied it to the following three problems.

Example 4.1. Consider the following linear system of fractional differential equations:

$$\begin{pmatrix} D_*^{\alpha_1} \\ D_*^{\alpha_2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad 0 < \alpha_1, \ \alpha_2 \leqslant 1, \tag{18}$$

subject to the initial conditions:

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{19}$$

The exact solution of this system, when $\alpha_1 = \alpha_2 = 1$, is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}.$$

In view of the homotopy analysis method presented above, if we select the auxiliary linear operators $\mathscr{L}_1 = \mathscr{L}_2 = \frac{d}{dt}$, the auxiliary functions $H_1(t) = H_2(t) = 1$, and the initial guesses $\begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we can construct the homotopy:

$$\begin{aligned} R_{1m}(\vec{x}_{m-1}(t)) &= D_*^{\alpha_1} x_{m-1}(t) - x_{m-1}(t) - y_{m-1}(t), \\ R_{2m}(\vec{y}_{m-1}(t)) &= D_*^{\alpha_2} y_{m-1}(t) + x_{m-1}(t) - y_{m-1}(t). \end{aligned} \tag{20}$$

Consequently, we have

$$\begin{aligned} x_m(t) &= \chi_m x_{m-1}(t) + \hbar_1 \int_0^t R_{1m}(\vec{x}_{m-1}(\tau)) d\tau, \\ y_m(t) &= \chi_m y_{m-1}(t) + \hbar_2 \int_0^t R_{2m}(\vec{y}_{m-1}(\tau)) d\tau. \end{aligned} \tag{21}$$

Solving the above linear systems of equations, the first few components of the homotopy analysis solution for Eq. (21) are derived as follows:

$$
\begin{aligned}
x_1 &= -\hbar_1 t,\\
y_1 &= -\hbar_2 t,\\
x_2 &= -\hbar_1 t + \frac{1}{2}\hbar_1(\hbar_1+\hbar_2)t^2 - \frac{\hbar_1^2}{\Gamma(3-\alpha_1)}t^{2-\alpha_1},\\
y_2 &= -\hbar_2 t + \frac{1}{2}\hbar_2(\hbar_2-\hbar_1)t^2 - \frac{\hbar_2^2}{\Gamma(3-\alpha_2)}t^{2-\alpha_2},\\
x_3 &= -\hbar_1 t + \hbar_1(\hbar_1+\hbar_2)t^2 - \frac{\hbar_1(\hbar_1^2+\hbar_2^2)}{6}t^3 - \frac{2\hbar_1^2}{\Gamma(3-\alpha_1)}t^{2-\alpha_1} + \frac{\hbar_1^2(2\hbar_1+\hbar_2)}{\Gamma(4-\alpha_1)}t^{3-\alpha_1} + \frac{\hbar_1\hbar_2^2}{\Gamma(4-\alpha_2)}t^{3-\alpha_2} - \frac{\hbar_1^3}{\Gamma(4-2\alpha_1)}t^{3-2\alpha_1},\\
y_3 &= -\hbar_2 t + \hbar_2(\hbar_2-\hbar_1)t^2 - \frac{2\hbar_2^2}{\Gamma(3-\alpha_2)}t^{2-\alpha_2} + \frac{\hbar_2(\hbar_1^2-\hbar_2^2+2\hbar_1\hbar_2)}{6}t^3 + \frac{\hbar_2^2(2\hbar_2-\hbar_1)}{\Gamma(4-\alpha_2)}t^{3-\alpha_2} - \frac{\hbar_1^2\hbar_2}{\Gamma(4-\alpha_1)}t^{3-\alpha_1}\\
&\quad - \frac{\hbar_2^3}{\Gamma(4-2\alpha_2)}t^{3-2\alpha_2}\\
&\vdots
\end{aligned}
\tag{22}
$$

Now, if we replace the auxiliary linear operator $\mathscr{L}_i = \frac{d}{dt}, i = 1,2$, in Eq. (11) by $\mathscr{L}_i = D_*^{\alpha_i}, i = 1,2$, and using Eq. (13) then we construct the homotopy as the follows:

$$
\begin{aligned}
x_m(t) &= \chi_m x_{m-1}(t) + \hbar_1 J^{\alpha_1}[R_{1m}(\vec{x}_{m-1}(t))],\\
y_m(t) &= \chi_m y_{m-1}(t) + \hbar_2 J^{\alpha_1}[R_{2m}(\vec{y}_{m-1}(t))].
\end{aligned}
\tag{23}
$$

Then we obtain the following:

$$
\begin{aligned}
x_1 &= \frac{-\hbar_1 t^{\alpha_1}}{\Gamma(1+\alpha_1)},\\
y_1 &= \frac{-\hbar_2 t^{\alpha_2}}{\Gamma(1+\alpha_2)},\\
x_2 &= \frac{-\hbar_1(1+\hbar_1)t^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\hbar_1\hbar_2 t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{\hbar_1^2 t^{2\alpha_1}}{\Gamma(1+2\alpha_1)},\\
y_2 &= \frac{-\hbar_2(1+\hbar_2)t^{\alpha_2}}{\Gamma(1+\alpha_2)} - \frac{\hbar_1\hbar_2 t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{\hbar_2^2 t^{2\alpha_2}}{\Gamma(1+2\alpha_2)},\\
x_3 &= \frac{-\hbar_1(1+\hbar_1)^2 t^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{2\hbar_1^2(1+\hbar_1)t^{2\alpha_1}}{\Gamma(1+2\alpha_1)} - \frac{\hbar_1^3 t^{3\alpha_1}}{\Gamma(1+3\alpha_1)} + \frac{\hbar_1\hbar_2(2+\hbar_1+\hbar_2)t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} - \frac{\hbar_1\hbar_2^2 t^{\alpha_1+2\alpha_2}}{\Gamma(1+\alpha_1+2\alpha_2)},\\
y_3 &= \frac{-\hbar_2(1+\hbar_2)t^{\alpha_2}}{\Gamma(1+\alpha_2)} + \frac{2\hbar_2^2(1+\hbar_2)t^{2\alpha_2}}{\Gamma(1+2\alpha_2)} - \frac{\hbar_2^3 t^{3\alpha_2}}{\Gamma(1+3\alpha_2)} - \frac{\hbar_1\hbar_2(2+\hbar_1+\hbar_2)t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{\hbar_2\hbar_1^2 t^{2\alpha_1+\alpha_2}}{\Gamma(1+2\alpha_1+\alpha_2)} + \frac{2\hbar_2^2\hbar_1 t^{\alpha_1+2\alpha_2}}{\Gamma(1+\alpha_1+2\alpha_2)},\\
&\vdots
\end{aligned}
\tag{24}
$$

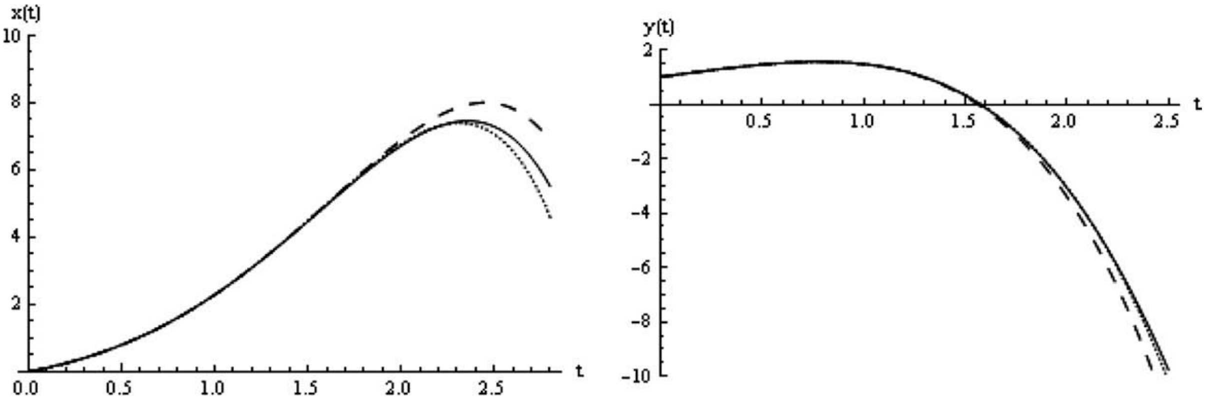

Fig. 1. Plots of system (18) when $\alpha_1 = \alpha_2 = 1$: *solid line*: *exact solution*; *dashed line*: $\hbar_1 = \hbar_2 = -1$, *dotted line*: $\hbar_1 = -1.4$, $\hbar_2 = -0.8$.

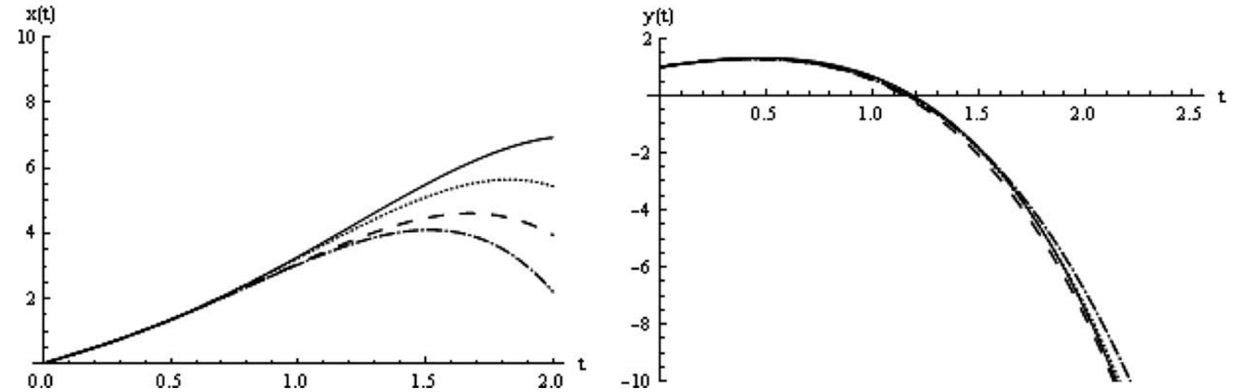

Fig. 2. Plots of system (18) when $\alpha_1 = 0.7$ and $\alpha_1 = 0.9$: *solid line*: $\hbar_1 = \hbar_2 = -1, \mathscr{L}_i = \frac{d}{dt}$; *dotted line*: $\hbar_1 = \hbar_2 = -1, \mathscr{L}_i = D_*^{\alpha_i}$; *dashed line*: $\hbar_1 = -1.4, \hbar_2 = -0.8, \mathscr{L}_i = \frac{d}{dt}$; *dash-dotted line*: $\hbar_1 = -1.4, \hbar_2 = -0.8, \mathscr{L}_i = D_*^{\alpha_i}$

Setting $\hbar_1 = \hbar_2 = -1$ in Eqs. (22) and (24), the above expressions are exactly the same as those given by the homotopy perturbation method and Adomian's decomposition method [11], respectively. This illustrates that the two methods are indeed special cases of the homotopy analysis method. However, the results given by the Adomian decomposition method and homotopy perturbation method converge to the corresponding numerical solutions in a rather small region, as shown in Fig. 1. But, different from those two methods, the homotopy analysis method provides us with a simple way to adjust and control the convergence region of series solution by choosing proper values for the auxiliary parameters $\hbar_1$ and $\hbar_2$ and by using the suitable auxiliary liner operators $\mathscr{L}_i = \frac{d}{dt}$, and $\mathscr{L}_i = D_*^{\alpha_i}$. It is to be noted that only four terms of the HAM series solution were used in evaluating the approximate solutions given in Figs. 1 and 2. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of $x(t)$ and $y(t)$.

Example 4.2. Consider the following nonlinear fractional system:

$$\begin{aligned} &D_*^{\alpha_1} x = \frac{1}{2}x, \\ &D_*^{\alpha_2} y = y + x^2, \quad 0 < \alpha_1, \alpha_2 \leqslant 1, \end{aligned} \tag{25}$$

subject to the initial condition:

$$\begin{aligned} &x(0) = 1, \\ &y(0) = 0. \end{aligned} \tag{26}$$

The exact solution of this system, when $\alpha_1 = \alpha_2 = 1$, is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{t/2} \\ te^t \end{pmatrix}.$$

(I) To solve the above system using HAM, we select the auxiliary linear operators $\mathscr{L}_1 = \mathscr{L}_2 = \frac{d}{dt}$, the auxiliary functions $H_1(t) = H_2(t) = 1$, and the initial guesses $\begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we can construct the homotopy as

$$\begin{aligned} &R_{1m}(\vec{x}_{m-1}(t)) = D_*^{\alpha_1} x_{m-1}(t) - \frac{1}{2}x_{m-1}(t), \\ &R_{2m}(\vec{y}_{m-1}(t)) = D_*^{\alpha_2} y_{m-1}(t) - y_{m-1}(t) - \sum_{i=0}^{m-1} x_i(t) x_{m-i-1}(t). \end{aligned} \tag{27}$$

Consequently, we have

$$\begin{aligned} &x_m(t) = \chi_m x_{m-1}(t) + \hbar_1 \int_0^t R_{1m}(\vec{x}_{m-1}(\tau)) d\tau, \\ &y_m(t) = \chi_m y_{m-1}(t) + \hbar_2 \int_0^t R_{2m}(\vec{y}_{m-1}(\tau)) d\tau. \end{aligned} \tag{28}$$

Solving the above system of equations, the first few components of the homotopy analysis solution for Eq. (28) are derived as follows:

$$\begin{aligned}
x_0 &= 1,\\
y_0 &= 0,\\
x_1 &= \frac{-\hbar_1}{2}t,\\
y_1 &= -\hbar_2 t,\\
x_2 &= \frac{-\hbar_1}{2}t + \frac{\hbar_1^2}{8}t^2 - \frac{\hbar_1^2}{2\Gamma(3-\alpha_1)}t^{2-\alpha_1},\\
y_2 &= -\hbar_2 t - \frac{\hbar_1^2+\hbar_2^2}{2}t^2 - \frac{\hbar_2^2}{\Gamma(3-\alpha_2)}t^{2-\alpha_2},\\
x_3 &= \frac{-\hbar_1}{2}t + \frac{\hbar_1^2}{4}t^2 - \frac{\hbar_1^3}{48}t^3 - \frac{\hbar_1^2}{\Gamma(3-\alpha_1)}t^{2-\alpha_1} + \frac{\hbar_1^3}{2\Gamma(4-\alpha_1)}t^{3-\alpha_1} - \frac{\hbar_1^3}{2\Gamma(4-2\alpha_1)}t^{3-2\alpha_1},\\
y_3 &= -\hbar_2 t + \frac{\hbar_1(\hbar_1+\hbar_2)+2\hbar_2^2}{2}t^2 - \frac{\hbar_2(2\hbar_1^2+\hbar_2^2)}{6}t^3 - \frac{2\hbar_2^2}{\Gamma(3-\alpha_2)}t^{2-\alpha_2} + \frac{\hbar_1^2\hbar_2}{\Gamma(4-\alpha_1)}t^{3-\alpha_1}\\
&\quad + \frac{\hbar_2(\hbar_1^2+2\hbar_2^2)}{\Gamma(4-\alpha_2)}t^{3-\alpha_2} - \frac{\hbar_2^3}{\Gamma(4-2\alpha_2)}t^{3-2\alpha_2}.
\end{aligned} \tag{29}$$

Also, if we replace the auxiliary linear operator $\mathscr{L}_i = \frac{d}{dt}, i = 1, 2$, in Eq. (11) by $\mathscr{L}_i = D_*^{\alpha_i}, i = 1, 2$, and using Eq. (13) then we construct the homotopy as follows:

$$\begin{aligned}
x_m(t) &= \chi_m x_{m-1}(t) + \hbar_1 J^{\alpha_1}[R_{1m}(\vec{x}_{m-1}(t))],\\
y_m(t) &= \chi_m y_{m-1}(t) + \hbar_2 J^{\alpha_1}[R_{2m}(\vec{y}_{m-1}(t))].
\end{aligned} \tag{30}$$

Then we obtain

$$\begin{aligned}
x_1 &= \frac{-\hbar_1 t^{\alpha_1}}{2\Gamma(1+\alpha_1)},\\
y_1 &= \frac{-\hbar_2 t^{\alpha_2}}{\Gamma(1+\alpha_2)},\\
x_2 &= \frac{-\hbar_1(1+\hbar_1)t^{\alpha_1}}{2\Gamma(1+\alpha_1)} + \frac{\hbar_1^2 t^{2\alpha_1}}{4\Gamma(1+2\alpha_1)},\\
y_2 &= \frac{-\hbar_2(1+\hbar_2)t^{\alpha_2}}{\Gamma(1+\alpha_2)} + \frac{\hbar_1\hbar_2 t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{\hbar_2^2 t^{2\alpha_2}}{\Gamma(1+2\alpha_2)},\\
x_3 &= \frac{-\hbar_1(1+\hbar_1)^2 t^{\alpha_1}}{2\Gamma(1+\alpha_1)} + \frac{\hbar_1^2(1+\hbar_1)t^{2\alpha_1}}{2\Gamma(1+2\alpha_1)} - \frac{\hbar_1^3 t^{3\alpha_1}}{8\Gamma(1+3\alpha_1)},\\
y_3 &= \frac{-\hbar_2(1+\hbar_2)^2 t^{\alpha_2}}{\Gamma(1+\alpha_2)} + \frac{2\hbar_2^2(1+\hbar_2)t^{2\alpha_2}}{\Gamma(1+2\alpha_2)} - \frac{\hbar_2^3 t^{3\alpha_2}}{\Gamma(1+3\alpha_2)} + \frac{\hbar_1\hbar_2(2+\hbar_1+\hbar_2)t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)}\\
&\quad - \frac{\hbar_1\hbar_2^2 t^{\alpha_1+2\alpha_2}}{\Gamma(1+\alpha_1+2\alpha_2)} - \frac{\hbar_1^2\hbar_2}{2}\left[\frac{1}{2\Gamma(1+\alpha_1)^2} + \frac{1}{\Gamma(1+2\alpha_1)}\right]\frac{\Gamma(1+2\alpha_2)}{\Gamma(1+2\alpha_1+\alpha_2)}t^{2\alpha_1+\alpha_2},\\
&\ \ \vdots
\end{aligned} \tag{31}$$

Setting $\hbar_1 = \hbar_2 = -1$ in Eqs. (29) and (31), the above expressions are exactly the same as those given by the homotopy perturbation method and Adomian's decomposition method, respectively.

(II) If we chose the auxiliary functions $H_1(t) = t^{0.2}$ and $H_2\ (t) = t^{0.7}$ in Eqs. (16) and the auxiliary linear operators $\mathscr{L}_i = \frac{d}{dt}, i = 1, 2$, then we find

$$\begin{aligned}
x_0 &= 1,\\
y_0 &= 0,\\
x_1 &= -0.42\hbar_1 t^{1.2},\\
y_1 &= -0.59\hbar_2 t^{1.7},\\
x_2 &= -0.42\hbar_1 t^{1.2} + \hbar_1^2\left[0.09t^{2.4} - \frac{0.46}{(2.4-\alpha_1)\Gamma(2.2-\alpha_1)}t^{2.4-\alpha_1}\right],\\
y_2 &= -0.59\hbar_2 t^{1.7} + 0.29\hbar_1\hbar_2 t^{2.9} + \hbar_2^2\left[0.17t^{3.4} - \frac{0.91}{(3.4-\alpha_2)\Gamma(2.7-\alpha_2)}t^{3.4-\alpha_2}\right],\\
&\ \ \vdots
\end{aligned} \tag{32}$$

If we replace the auxiliary linear operator $\mathscr{L}_i = \frac{d}{dt}, i = 1, 2$, by $\mathscr{L}_i = D_*^{\alpha_i}, i = 1, 2$, in the above expressions, we get the following series solutions:

$$\begin{aligned}
x_0 &= 1,\\
y_0 &= 0,\\
x_1 &= -\frac{0.46\hbar_1}{\Gamma(\alpha_1+1.2)}t^{\alpha_1+0.2},\\
y_1 &= -\frac{0.91\hbar_2}{\Gamma(\alpha_2+1.7)}t^{\alpha_2+0.7},\\
x_2 &= -\frac{0.46\hbar_1}{\Gamma(\alpha_1+1.2)}t^{\alpha_1+0.2} + \hbar_1^2\left[\frac{0.23\Gamma(\alpha_1+1.4)}{\Gamma(\alpha_1+1.2)\Gamma(2\alpha_1+1.4)}t^{2\alpha_1+0.4} - \frac{0.44}{\Gamma(\alpha_1+1.4)}t^{\alpha_1+0.4}\right],\\
y_2 &= -\frac{0.91\hbar_2}{\Gamma(\alpha_2+1.7)}t^{\alpha_2+0.7f} + \frac{0.92\hbar_1\hbar_2\Gamma(\alpha_1+1.9)}{\Gamma(\alpha_1+1.2)\Gamma(\alpha_1+\alpha_2+1.9)}t^{\alpha_1+\alpha_2+0.9}\\
&\quad + \hbar_2^2\left[\frac{0.91\Gamma(\alpha_2+2.4)}{\Gamma(\alpha_2+1.7)\Gamma(2\alpha_2+2.4)}t^{2\alpha_2+1.4} - \frac{1.24}{\Gamma(\alpha_2+2.4)}t^{\alpha_2+1.4}\right],\\
&\vdots
\end{aligned} \tag{33}$$

Figs. 3 and 4 show the approximate solutions for Eq. (25) obtained for different values of α_1, α_2, $\hbar_1$, $\hbar_2$, $H_1(t)$ and $H_2(t)$ using HAM. From the numerical results in Figs. 3 and 4, we can adjust and control the convergence region of series solution by choosing proper values for the auxiliary parameters $\hbar_1$,$\hbar_2$ and the auxiliary functions $H_1(t)$, $H_2(t)$ and by using the suitable auxiliary liner operators $\mathscr{L}_i = \frac{d}{dt}$ and $\mathscr{L}_i = D_*^{\alpha_i}$. Fig. 3 shows that the best solution results when we use the auxiliary parameters $\hbar_1 = \hbar_2 = -1.5$ and auxiliary functions $H_1(t) = t^{0.2}$, $H_2(t) = t^{0.7}$ and $\mathscr{L}_i = D_*^{\alpha_i}$. Fig. 4 shows a good agreement between the

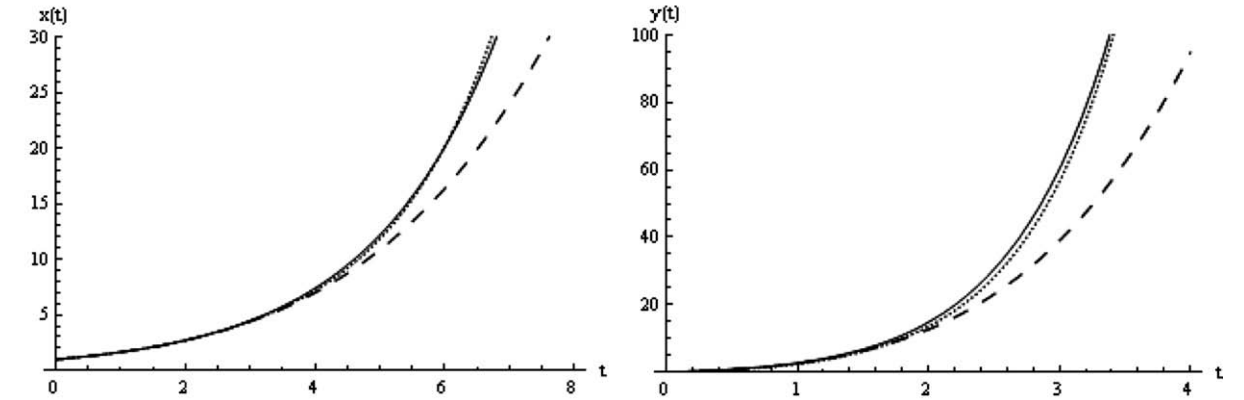

Fig. 3. Plots of system (25) when $\alpha_1 = \alpha_2 = 1$: *solid line*: *exact; dashed line*: $\hbar_1 = \hbar_2 = -1$, $H_1(t) = H_2(t) = 1$; *dotted line*: $\hbar_1 = \hbar_2 = -1.5$, $H_1(t) = t^{0.2}$, $H_2(t) = t^{0.7}$.

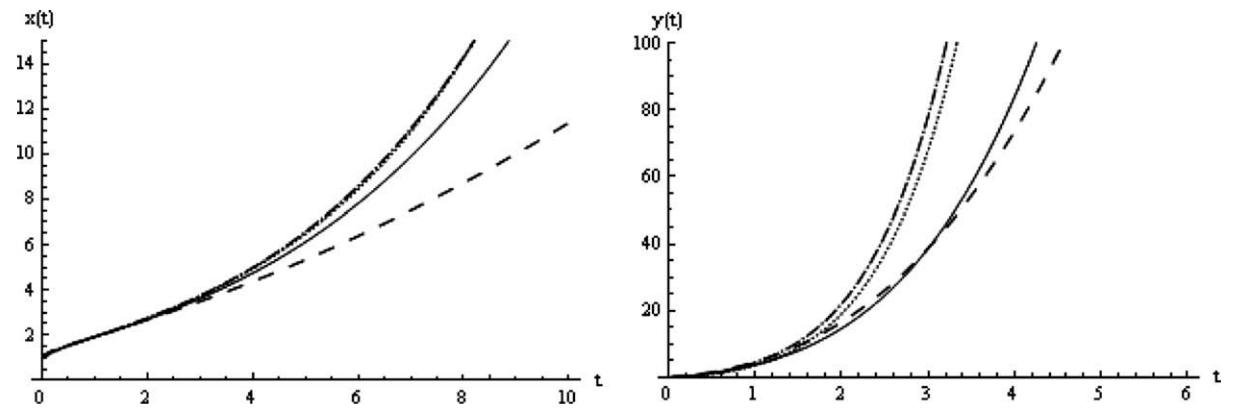

Fig. 4. Plots of system (25) when $\alpha_1 = 0.5$ and $\alpha_1 = 0.8$: *solid line*: $\hbar_1 = \hbar_2 = -1, H_1(t) = H_2(t) = 1, \mathscr{L}_i = \frac{d}{dt}$; *dashed line*: $\hbar_1 = \hbar_2 = -1, H_1(t) = 1, H_2(t) = 1, \mathscr{L}_i = D_*^{\alpha_i}$; *dotted line*: $\hbar_1 = \hbar_2 = -1.5, H_1(t) = t^{0.2}, H_2(t) = t^{0.7}, \mathscr{L}_i = \frac{d}{dt}$; *dash-dotted line*: $\hbar_1 = \hbar_2 = -1.5, H_1(t) = t^{0.2}, H_2(t) = t^{0.7}, \mathscr{L}_i = D_*^{\alpha_i}$.

solutions obtained using the auxiliary liner operators $\mathscr{L} = \frac{d}{dt}$ and $\mathscr{L} = D_*^{\alpha_i}$ in the case of using the fractional derivatives $\alpha_1 = 0.5$ and $\alpha_2 = 0.8$.

Example 4.3. Consider the nonlinear system of fractional differential equations:

$$\begin{aligned} &D_*^{\alpha_1} x = x,\\ &D_*^{\alpha_2} y = 2x^2,\\ &D_*^{\alpha_3} z = 3xy, \end{aligned} \tag{34}$$

subject to the initial conditions:

$$x(0) = 1, \quad y(0) = 1, \quad z(0) = 0. \tag{35}$$

The exact solution of this system, when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} e^t \\ e^{2t} \\ e^{3t} - 1 \end{pmatrix}.$$

If we select the auxiliary linear operators $\mathscr{L}_1 = \mathscr{L}_2 = \mathscr{L}_3 = \frac{d}{dt}$, the auxiliary functions $H_1(t) = H_2(t) = H_3(t) = 1$, and the initial guesses $x_0(t) = 1$, $y_0(t) = 1$, $z_0\,(t) = 0$, we can construct the homotopy:

$$\begin{aligned} x_m(t) &= \chi_m x_{m-1}(t) + \hbar_1 \int_0^t R_{1m}(\vec{x}_{m-1}(\tau))d\tau,\\ y_m(t) &= \chi_m y_{m-1}(t) + \hbar_2 \int_0^t R_{2m}(\vec{y}_{m-1}(\tau))d\tau,\\ z_m(t) &= \chi_m z_{m-1}(t) + \hbar_3 \int_0^t R_{3m}(\vec{z}_{m-1}(\tau))d\tau, \end{aligned} \tag{36}$$

where

$$\begin{aligned} R_{1m}(\vec{x}_{m-1}(t)) &= D_*^{\alpha_1} x_{m-=1}(t) - x_{m-=1}(t),\\ R_{2m}(\vec{y}_{m-1}(t)) &= D_*^{\alpha_1} y_{m-=1}(t) - 2\sum_{i=0}^{m-1} x_i(t)x_{m-1-i}(t),\\ R_{3m}(\vec{z}_{m-1}(t)) &= D_*^{\alpha_1} z_{m-=1}(t) - 3\sum_{i=0}^{m-1} x_i(t)y_{m-1-i}(t). \end{aligned} \tag{37}$$

The first few components of the homotopy analysis solution for Eq. (36) are derived as follows:

$$\begin{aligned} x_1 &= -\hbar_1 t,\\ y_1 &= -2\hbar_2 t,\\ z_1 &= -3\hbar_3 t,\\ x_2 &= -\hbar_1 t + \frac{\hbar_1^2}{2}t^2 - \frac{\hbar_1^2}{\Gamma(3-\alpha_1)}t^{2-\alpha_1},\\ y_2 &= -2\hbar_2 t + 2\hbar_1\hbar_2 t^2 - \frac{2\hbar_2^2}{\Gamma(3-\alpha_2)}t^{2-\alpha_2},\\ z_2 &= -3\hbar_3 t + \frac{3\hbar_3(\hbar_1+2\hbar_2)}{2}t^2 - \frac{3\hbar_3^2}{\Gamma(3-\alpha_3)}t^{2-\alpha_3},\\ x_3 &= -\hbar_1 t + \hbar_1^2 t^2 - \frac{\hbar_1^3}{6}t^3 - \frac{2\hbar_1^2}{\Gamma(3-\alpha_1)}t^{2-\alpha_1} + \frac{2\hbar_1^3}{\Gamma(4-\alpha_1)}t^{3-\alpha_1} - \frac{\hbar_1^3}{\Gamma(4-2\alpha_1)}t^{3-2\alpha_1},\\ y_3 &= -2\hbar_2 t + 4\hbar_1\hbar_2 t^2 - \frac{4\hbar_2\hbar_1^2}{3}t^3 - \frac{4\hbar_2^2}{\Gamma(3-\alpha_2)}t^{2-\alpha_2} + \frac{4\hbar_1\hbar_2(\hbar_1+\hbar_2)}{\Gamma(4-\alpha_2)}t^{3-\alpha_2} - \frac{2\hbar_2^3}{\Gamma(4-2\alpha_2)}t^{3-2\alpha_2},\\ z_3 &= -3\hbar_3 t + 3\hbar_3(\hbar_1+2\hbar_2)t^2 - \frac{\hbar_1\hbar_3(\hbar_1+8\hbar_2)}{2}t^3 - \frac{6\hbar_3^2}{\Gamma(3-\alpha_3)}t^{2-\alpha_3} + \frac{3\hbar_1^2\hbar_3}{\Gamma(4-\alpha_1)}t^{3-\alpha_1}\\ &\quad + \frac{6\hbar_2^2\hbar_3}{\Gamma(4-\alpha_2)}t^{3-\alpha_2} + \frac{3\hbar_3^2(\hbar_1+2\hbar_2)}{\Gamma(4-\alpha_3)}t^{3-\alpha_3} - \frac{3\hbar_3^3}{\Gamma(4-2\alpha_3)}t^{3-2\alpha_3},\\ &\ \vdots \end{aligned} \tag{38}$$

Now, if we replace the auxiliary linear operator $\mathscr{L}_i = \frac{d}{dt}, i = 1,2,3$, in Eq. (11) by $\mathscr{L}_i = D_*^{\alpha_i}, i = 1,2,3$, and using Eq. (13) then we construct the following homotopy:

$$\begin{aligned} x_m(t) &= \chi_m x_{m-1}(t) + \hbar_1 J^{\alpha_1}[R_{1m}(\vec{x}_{m-1}(t))],\\ y_m(t) &= \chi_m y_{m-1}(t) + \hbar_2 J^{\alpha_2}[R_{2m}(\vec{y}_{m-1}(t))],\\ z_m(t) &= \chi_m z_{m-1}(t) + \hbar_3 J^{\alpha_3}[R_{3m}(\vec{z}_{m-1}(t))]. \end{aligned} \tag{39}$$

Then we find

$$
\begin{aligned}
x_1 &= \frac{-\hbar_1 t^{\alpha_1}}{\Gamma(1+\alpha_1)},\\
y_1 &= \frac{-2\hbar_2 t^{\alpha_2}}{\Gamma(1+\alpha_2)},\\
z_1 &= \frac{-3\hbar_3 t^{\alpha_3}}{\Gamma(1+\alpha_3)},\\
x_2 &= \frac{-\hbar_1(1+\hbar_1)t^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\hbar_1^2 t^{2\alpha_1}}{\Gamma(1+2\alpha_1)},\\
y_2 &= \frac{-2\hbar_2(1+\hbar_2)t^{\alpha_2}}{\Gamma(1+\alpha_2)} + \frac{4\hbar_1\hbar_2 t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)},\\
z_2 &= \frac{-3\hbar_3(1+\hbar_3)t^{\alpha_3}}{\Gamma(1+\alpha_3)} + \frac{6\hbar_2\hbar_3 t^{\alpha_2+\alpha_3}}{\Gamma(1+\alpha_2+\alpha_3)} + \frac{3\hbar_1\hbar_3 t^{\alpha_1+\alpha_3}}{\Gamma(1+\alpha_1+\alpha_3)},\\
x_3 &= \frac{-\hbar_1(1+\hbar_1)^2 t^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{2\hbar_1^2(1+\hbar_1)t^{2\alpha_1}}{\Gamma(1+2\alpha_1)} - \frac{\hbar_1^3 t^{3\alpha_1}}{\Gamma(1+3\alpha_1)},\\
y_3 &= \frac{-2\hbar_2(1+\hbar_2)t^{\alpha_2}}{\Gamma(1+\alpha_2)} + \frac{4\hbar_1\hbar_2(2+\hbar_1+\hbar_2)t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} - 2\hbar_1^2\hbar_2\left[\frac{1}{\Gamma(1+\alpha_1)^2} + \frac{2}{\Gamma(1+2\alpha_1)}\right]\frac{\Gamma(1+2\alpha_1)}{\Gamma(1+2\alpha_1+\alpha_2)} t^{2\alpha_{1}+\alpha_2},\\
z_3 &= \frac{-3\hbar_3(1+\hbar_3)^2 t^{\alpha_3}}{\Gamma(1+\alpha_3)} + \frac{3\hbar_1\hbar_3(2+\hbar_1+\hbar_3)t^{\alpha_1+\alpha_3}}{\Gamma(1+\alpha_1+\alpha_3)} + \frac{6\hbar_2\hbar_3(2+\hbar_2+\hbar_3)t^{\alpha_2+\alpha_3}}{\Gamma(1+\alpha_2+\alpha_3)}\\
&\quad - 6\hbar_1\hbar_2\hbar_3\left[\frac{1}{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)} + \frac{2}{\Gamma(1+\alpha_1+\alpha_2)}\right]\frac{\Gamma(1+\alpha_1+\alpha_2)}{\Gamma(1+\alpha_1+\alpha_2+\alpha_3)} t^{\alpha_1+\alpha_2+\alpha_3} - \frac{3\hbar_1^2\hbar_3 t^{\alpha_3+2\alpha_1}}{\Gamma(1+\alpha_3+2\alpha_1)},\\
&\vdots
\end{aligned}
\tag{40}
$$

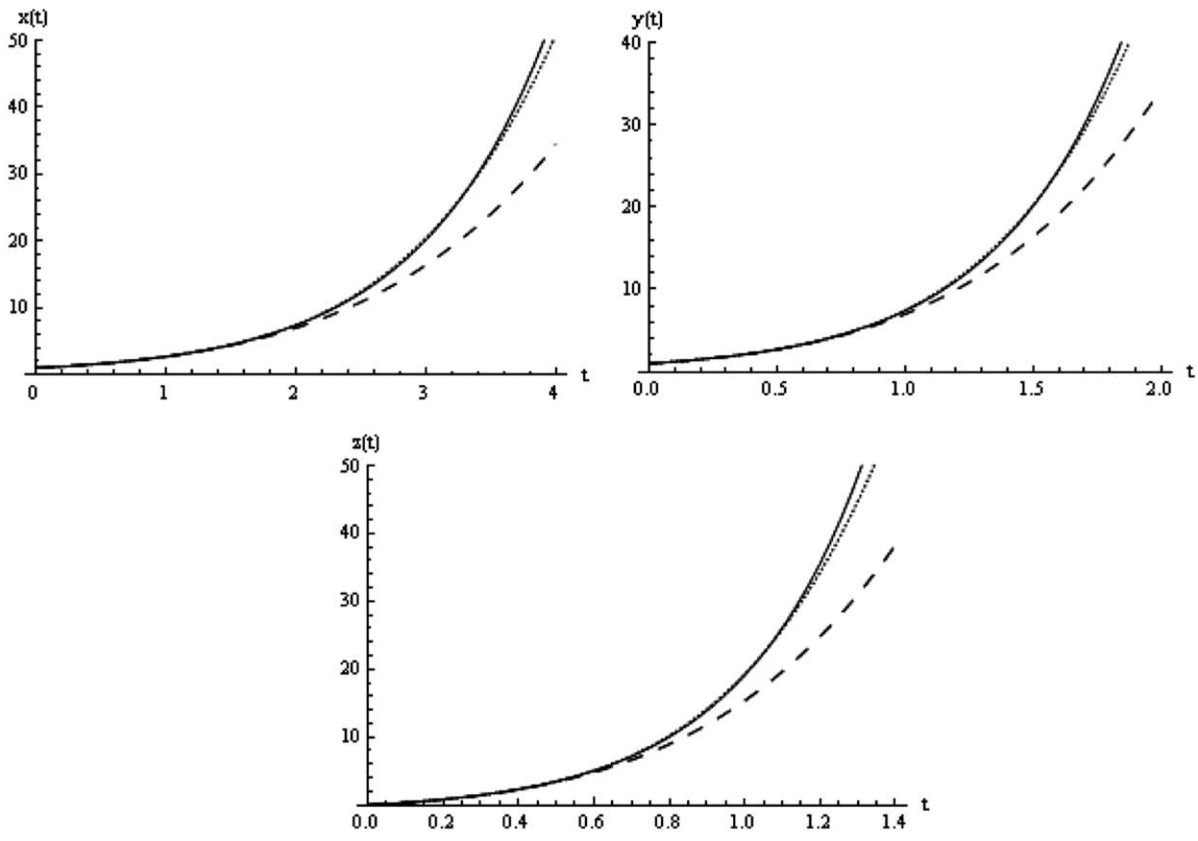

Fig. 5. Plot of system (34) when $\alpha_1 = \alpha_2 = \alpha_3 = 1$: *solid line*: *exact*; *dotted line*: $\hbar_1 = -1.4$, $\hbar_2 = -1.2$, $\hbar_3 = -1.5$; *dashed line*: $\hbar_1 = \hbar_2 = \hbar_3 = -1$.

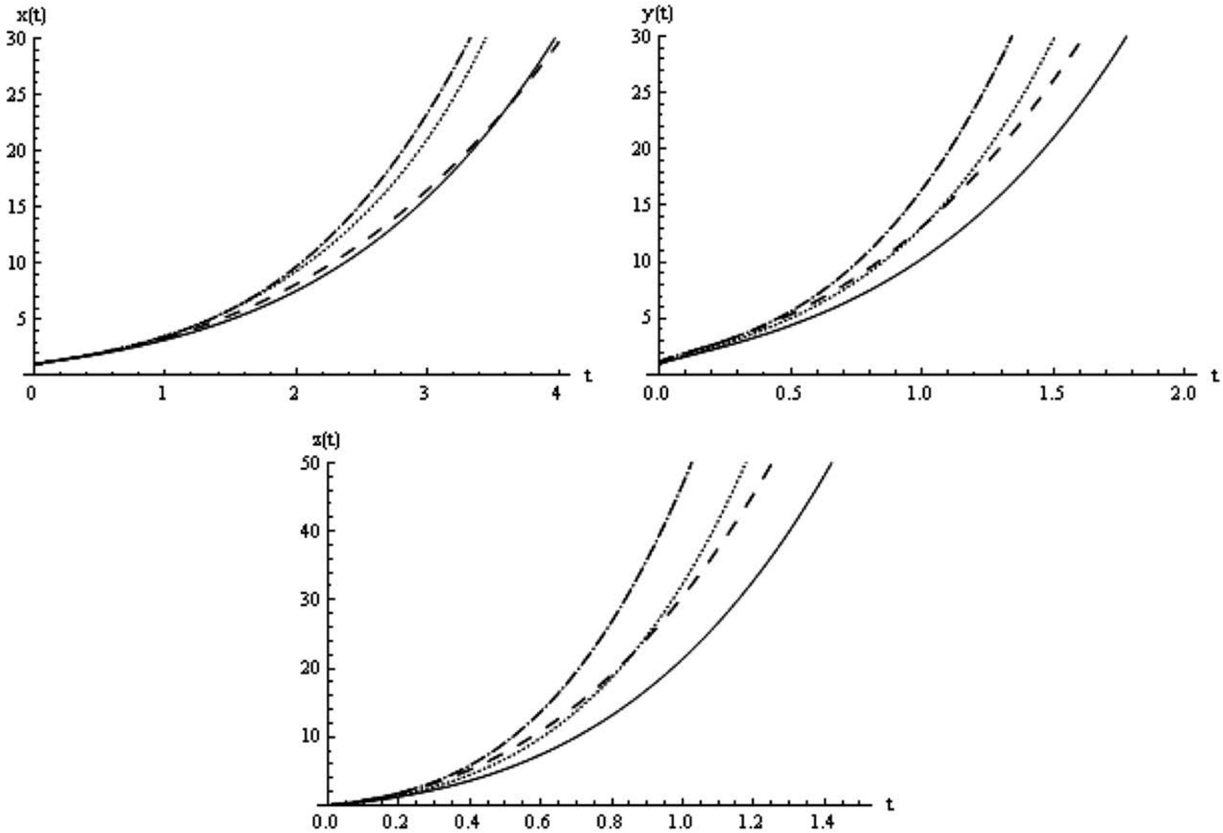

Fig. 6. Plot of system (34) when $\alpha_1 = 0.75$, $\alpha_2 = 0.5$ and $\alpha_3 = 0.9$: *solid line*: $\hbar_1 = \hbar_2 = \hbar_3 = -1, \mathscr{L}_i = \frac{d}{dt}$; *dashed line*: $\hbar_1 = \hbar_2 = \hbar_3 = -1, \mathscr{L}_i = D_*^{\alpha_i}$; *dotted line*: $\hbar_1 = -1.4, \hbar_2 = -1.2, \hbar_3 = -1.5, \mathscr{L}_i = \frac{d}{dt}$, *dash-dotted line*: $\hbar_1 = -1.4, \hbar_2 = -1.2, \hbar_3 = -1.5, \mathscr{L}_i = D_*^{\alpha_i}$.

Similarly, the results of Adomian decomposition method and homotopy perturbation method, which correspond to the HAM solution in the special case of $\hbar_1 = \hbar_2 = \hbar_3 = -1$, are valid in a rather small region, as shown in Fig. 5. Figs. 5 and 6 show the approximate solutions for Eq. (34) obtained for different values of α_1, α_2, α_3, $\hbar_1$, $\hbar_2$, and $\hbar_3$ using HAM.

5. Conclusions

The homotopy analysis method is used for calculating approximate solutions for systems of fractional differential equations. Different from all other analytic methods, it provides us with a simple way to adjust and control the convergence region of solution series by choosing proper values for auxiliary parameter $\hbar$, auxiliary function $H(t)$, and auxiliary linear operator $\mathscr{L}$. Also, we showed that the homotopy perturbation method and Adomain decomposition method are special cases of the homotopy analysis method.

There are some important points to make here. First, we have great freedom to choose the auxiliary parameter $\hbar$, auxiliary function $H(t)$, auxiliary linear operator $\mathscr{L}$, and the initial guesses. Second, the HAM was shown to be a simple, yet powerful analytic–numeric scheme for handling for systems of fractional differential equations. Finally, generally speaking, the proposed approach can be further implemented to solve other nonlinear problems in fractional calculus field.

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