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Square-root dynamics of a giving up smoking model

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ABSTRACT

The paper presents a new giving up smoking model for which interaction term is square root of potential and occasional smokers of model presented in Zaman (2011) [15]. First, we will show formulation of the model. Then we will discuss local and global stability of the model and its general solutions. The non-standard finite difference method (NSFD) is used to solve the new giving up smoking model. Both non-negativity and conservative law for differential equations system are discussed. Numerical results are presented graphically and compared well with those obtained by Runge–Kutta fourth-order method (RK4) and ODE45.

1. Introduction

Smoking is known to be the biggest cause of both preventable and premature death, not only in the US, but also worldwide. Smoking-related diseases are cause of over 440,000 deaths in the US annually and for UK this figure stands over 105,000 annually [1]. The life expectancy of smoker is cut short by 10–12 years and more than half of all smokers die from smoking-related diseases. Smoking is also known as slow killer. Comparative smoking facts show that the risk of heart attack is 70% higher among smokers than among non-smokers. The incidence of lung cancer is ten times greater in smokers than non-smokers and one out of ten people that smoke will die from this disease. Some 80% of smokers will at one time be diagnosed with heart disease, emphysema or chronic bronchitis. Of the diseases attributable to tobacco habit, 29% are from lung cancer and 24% are caused by heart disease [2]. Some other cancers have also been linked to smoking, including throat, mouth, stomach, cervix, breast and pancreas cancer. All these harmful diseases are caused by smoking, because a cigarette contain over 4000 chemical compounds and toxins.

Mathematical modeling is playing an important role in spread and control of many diseases including smoking [3–10]. Literature of *SIR* disease transmission model is quite large and has been studied by many authors [11]. In the *SIR* model, S stands for number of individuals that are susceptible to infection, I stands for number of individuals who are infectious and R denotes number of individuals who have recovered [3]. Similarly, many authors have studied the SEIR, SIRC and SIS models for many diseases. The studies of modeling have shown that incidence rate plays an important role in the epidemic models. In many models, bilinear incidence rate is studied and some authors have introduced the saturated incidence rate in different models, where saturated rate is more reasonable than bilinear incidence rate because it includes the behaviorial change and crowding effect of the infective individual [12]. Beside this, Mickens [13] has introduced the square root dynamic in model. In his work, he has constructed the path of trajectories in the two-dim S-I phase space. The major reason for square root model is the possibility of having population go to extinction in a finite time. Mathematical models of smoking, with

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both linear and nonlinear incidence rates have been discussed by many authors. Castillo-Garsow et al. [14] have proposed a simple mathematical model for giving up smoking. They consider a system with total constant population which is divided into three classes: potential smokers (P), smokers (S), and quit smokers (Q). Sharomi and Gumel [8] have developed mathematical model by introducing mild and chain classes. They have presented the development and public health impact of smoking related illnesses. Several more authors have done a lot of work in order to understand dynamics of smoking see for example [2,8,14,15]. The dynamical interaction of giving-up smoking model is presented in [15]. However, no one has used the square root interaction in giving-up smoking model.

In order to illustrate this, we will take a model with constant birth rate λ for the potential smokers individual. The complete dynamics is given by

$$\frac{dP}{dt} = \lambda - \beta f(L,P) - (d+\mu)P, \tag{1}$$

$$\frac{dL}{dt} = \beta f(L,P) - (\gamma + d + \mu)L, \tag{2}$$

$$\frac{dS}{dt} = \gamma L - (\delta + d + \mu)S, \tag{3}$$

$$\frac{dQ}{dt} = \delta S - (\mu + d)Q, \tag{4}$$

where $P(t), L(t), S(t)$ and $Q(t)$ denote numbers of potential smokers, occasional smokers, smokers and quit smokers at time t, respectively. Here μ is natural death rate, γ is recover rate from infection, β is the transmission coefficient, δ is quit rate of smoking, d represents death rate for potential smokers, occasional smoker, smoker and quit smoker related to smoking disease.

In this paper, we consider the interaction of potential and occasional smokers in the form

$$f(P,L) = \sqrt{PL}. \tag{5}$$

We focus here on the construction of a discrete-time model for the above system. It should be noted that the system of ODE's consisting of Eqs. (1)–(5) satisfy the conservation law

$$\frac{dN}{dt} = \lambda - (\mu + d)N, \tag{6}$$

where $N(t)$ is total population size at time t with

$$N(t) = P(t) + L(t) + S(t) + Q(t). \tag{7}$$

This equation has the exact solution

$$N(t) = \frac{\lambda}{(\mu+d)} + \left(N_0 - \frac{\lambda}{(\mu+d)}\right)e^{-(\mu+d)t}, \tag{8}$$

with

$$N(0) = P(0) + L(0) + S(0) + Q(0). \tag{9}$$

Also we have

$$P(0) \geqslant 0,\ L(0) \geqslant 0,\ S(0) \geqslant 0,\ Q(0) \geqslant 0 \Rightarrow P(t) \geqslant 0,\ L(t) \geqslant 0,\ S(t) \geqslant 0,\ Q(t) \geqslant 0. \tag{10}$$

So the solution has positivity property. By using Eq. (5) in the system (1)–(4), we obtain

$$\frac{dp}{dt} = \lambda - \beta\sqrt{PL} - (d+\mu)P, \tag{11}$$

$$\frac{dL}{dt} = \beta\sqrt{PL} - (\gamma + d + \mu)L, \tag{12}$$

$$\frac{dS}{dt} = \gamma L - (\delta + d + \mu)S, \tag{13}$$

$$\frac{dQ}{dt} = \delta S - (\mu + d)Q, \tag{14}$$

along with the conservative law

$$\frac{dN}{dt} = \lambda - (\mu + d)N, \tag{15}$$

$$N(t) = P(t) + L(t) + S(t) + Q(t). \tag{16}$$

2. Stability analysis of model

The system of ODE's given by (11)–(15) has unique non-trivial solution. By setting the right hand side of the Eqs. (11)–(15) equal to zero, we get

$$P^* = \frac{\lambda(\mu+\gamma+d)}{(\beta^2+(\mu+d)(\mu+\gamma+d))}, \quad (17)$$

$$L^* = \left(\frac{\beta}{\mu+\gamma+d}\right)^2 P^*, \quad (18)$$

$$S^* = \frac{\gamma}{(\delta+\mu+d)}\left(\frac{\beta}{\mu+\gamma+d}\right)^2 P^*, \quad (19)$$

$$Q^* = \frac{\gamma\delta}{(\mu+d)(\delta+\mu+d)}\left(\frac{\beta}{\mu+\gamma+d}\right)^2 P^*. \quad (20)$$

All the parameters are taken to be positive, then P^*, L^*, S^*, Q^* are positive. For the unique positive equilibria the Jacobian matrix at this fixed point is as

$$J(P^*, L^*, S^*, Q^*) = \begin{pmatrix} -a & -b & 0 & 0 \\ c & -b & 0 & 0 \\ 0 & \gamma & -d^* & 0 \\ 0 & 0 & \delta & -e \end{pmatrix},$$

where

$$a = \frac{2(\mu+d)(\mu+\gamma+d)+\beta^2}{2(\mu+\gamma+d)},$$
$$b = \frac{\mu+\gamma+d}{2},$$
$$c = \frac{\beta^2}{2(\mu+\gamma+d)},$$
$$d^* = \mu+\delta+d,$$
$$e = (d+\mu).$$

The eigenvalues $(\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4)$ are given by

$$\det[J(P^*, L^*, S^*, Q^*) - \lambda I_4] = 0,$$

where I_4 is the unit matrix of order 4×4. By evaluating, this determinant we obtain the following equation

$$(\lambda+d^*)(\lambda+e)((\lambda+a)(\lambda+b)+bc) = 0. \quad (21)$$

It is clear that $\lambda_1 = -d^*$ and $\lambda_2 = -e$ both are negative. After these two roots the Eq. (21) becomes

$$\lambda^2 + (a+b)\lambda + ab + bc = 0. \quad (22)$$

Let the remaining roots of this equation be λ_3, λ_4 and satisfying the following relations

$$\lambda_3\lambda_4 = ab+bc > 0, \quad \lambda_3+\lambda_4 = -ab-bc < 0. \quad (23)$$

From this we conclude that:

(i) If λ_3 and λ_4 are real, then both roots have same sign.
(ii) If λ_3 and λ_4 are real, then both roots are negative.
(iii) If λ_3 and λ_4 are complex, then $\lambda_3 = \lambda_4$ and the real parts are negative.
(iv) Thus, all the eigenvalues are negative or have negative real parts, and hence we conclude that this fixed point is located at (P^*, L^*, S^*, Q^*) is locally stable.

Theorem 2.1. *If the potential smoker does not reproduce i.e.,* $\lambda = 0$. *Then the square root interaction in giving-up smoking model is globally asymptotically stable.*

Proof. For the global stability of the giving-up smoking model we define the Lyapunov function in the following

$$V(t) = W_1(P-P^*) + W_2(L-L^*) + W_3S + W_4Q$$

where W_1, W_2, W_3, and W_4 are positive constants to be chosen later. Taking the time derivative and using Eqs. (1)–(4) we obtain

$$V'(t) = W_1[\lambda - \beta\sqrt{PL} - (d+\mu)P] + W_2[\beta\sqrt{PL} - (\gamma + d + \mu)L] + W_3[\gamma L - (\delta + d + \mu)S] + W_4[\delta S - (\mu + d)Q],$$

where ($\prime$) denotes the derivative with respect to time. By choosing the constants $W_1 = W_2 = W_3 = W_4 = 1$ and $\lambda = 0$ we get

$$V'(t) = -[(d+\mu)P + (d+\mu)L + (d+\mu)S + (\mu+d)Q].$$

Thus, the square root dynamics in the giving-up smoking model is globally asymptotically stable. This completes the proof. □

Now we study an interesting special case of the system when potential smoker does not reproduce i.e., $\lambda = 0$. The model becomes

$$\frac{dP}{dt} = -\beta\sqrt{PL} - (d+\mu)P, \tag{24}$$

$$\frac{dL}{dt} = \beta\sqrt{PL} - (\gamma + d + \mu)L, \tag{25}$$

$$\frac{dS}{dt} = \gamma L - (\delta + d + \mu)S, \tag{26}$$

$$\frac{dQ}{dt} = \delta S - (\mu + d)Q, \tag{27}$$

$$\frac{dN}{dt} = -(\mu + d)N. \tag{28}$$

Since the equations for $P(t)$ and $L(t)$ do not contain the third and fourth variables, $S(t)$ and $Q(t)$, we can show that Eqs. (24) and (25) can be solved exactly. To do so, we change variables as given

$$u(t) = \sqrt{P(t)}, \quad v(t) = \sqrt{L(t)}.$$

By using these variables, we obtain the following system of first order linear ordinary differential equations

$$\frac{du}{dt} = -\left(\frac{\beta}{2}\right)v - (d+\mu)u, \tag{29}$$

$$\frac{dv}{dt} = \left(\frac{\beta}{2}\right)u - (d + \mu + \gamma)v. \tag{30}$$

The exact solutions can be determined by using standard methods from elementary differential equations [16–18]. Note that for this case, all populations approach zero, i.e.,

$$Lim_{t\to\infty}\begin{pmatrix} P(t) \\ L(t) \\ S(t) \\ Q(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{31}$$

Finally, if the variable $v(t)$ is eliminated, then the following second-order, linear ODE is obtained

$$\frac{d^2u}{dt^2} + (2d + 2\mu + \gamma)\frac{du}{dt} + \left(\frac{4(d+\mu) - \beta^2}{4}\right)u = 0. \tag{32}$$

This is a linear, damped harmonic oscillator equation [17,18] having solution

$$u = e^{mt},$$

where

$$m = \frac{-(2(\mu+d)+\gamma) \pm \sqrt{(2(\mu+d)+\gamma)^2 - (4(d+\mu) - \beta^2)}}{2}.$$

3. The NSFD scheme

In general, the non-standard finite difference rules, introduced by Mickens [16,19,20], do not lead to unique discrete model for the solution of any dynamical system based on differential equations. First, we give basic rules of non-standard ordinary differential equations (ODE's) in the form:

$$y'_k = f(t, y_1, y_2, \ldots, y_m), \quad k = 1, 2, \ldots, m,$$

where $f(t, y_k(t))$ is the nonlinear term in differential equation. Using finite difference method we have

$$y'_1 = \frac{y_{1,k+1} - y_{1,k}}{\phi_k(h)}, \tag{33}$$

$$y'_2 = \frac{y_{2,k+1} - y_{2,k}}{\phi_k(h)}, \tag{34}$$

$$\vdots$$

$$y'_m = \frac{y_{m,k+1} - y_{m,k}}{\phi_k(h)}, \tag{35}$$

where ϕ_k is a function of step size $h = \Delta t$. The function ϕ_k have the following properties:

$$\phi_k(h) = h + o(h^2), \quad \text{for } h \to 0. \tag{36}$$

Examples of functions $\phi_k(h)$ that satisfy (36) are h, $\sin(h)$, $\sinh(h), e^h - 1$, $\frac{1-e^{-\lambda h}}{\lambda}$.

Non-linear terms can in general be replaced by nonlocal discrete representations, for example,

$$y^2 \approx y_k y_{k+1}, \tag{37}$$

$$y^3 \approx \left(\frac{y_{k+1} + y_{k-1}}{2}\right) y_k^2, \tag{38}$$

where $h = T/N, t_n = nh, n = 0, 1, \ldots, N \in Z^+$.

Now, we apply the NSFD to obtain numerical solution for giving up smoking model (11)–(15). For the discretization, we use the notation:

$$t \to t_k = (\Delta t)k, \quad (P(t), L(t), S(t), Q(t) \to P_k, L_k, S_k, Q_k),$$

where $\Delta t = h$ is the step-size.

The NSFD scheme for Eqs. (11)–(15) are given below shows non-negativity, unique fixed-point similar to ODE's system, and satisfying the same conservative law

$$\frac{P_{k+1} - P_k}{\phi} = \lambda - \beta\sqrt{L_k P_{k+1}} - (d + \mu)P_{k+1}, \tag{39}$$

$$\frac{L_{k+1} - L_k}{\phi} = \beta\sqrt{L_{k+1} P_k} - (\gamma + d + \mu)L_{k+1}, \tag{40}$$

$$\frac{S_{k+1} - S_k}{\phi} = \gamma L_{k+1} - (\delta + d + \mu)S_{k+1}, \tag{41}$$

$$\frac{Q_{k+1} - Q_k}{\phi} = \delta S_{k+1} - (\mu + d), \tag{42}$$

$$\frac{N_{k+1} - N_k}{\phi} = \lambda - (\mu + d)N_{k+1}. \tag{43}$$

Here

$$N_k = P_k + L_k + S_k + Q_k,$$

and the denominator function [12]$\phi = \phi(h, \vartheta)$, where $\vartheta = \mu + d$ is

$$\phi(h, \vartheta) = \frac{e^{\vartheta h} - 1}{\vartheta}.$$

Note, that the NSFD scheme, given in Eq. (43) is exact scheme for conservation law, by Eq. (15) [20].

The above system of difference equations has fixed-points that can be calculated by replacing (P_k, L_k, S_k, Q_k) by constant values $(\overline{P}, \overline{L}, \overline{S}, \overline{Q})$. An easy, direct evaluation of the resulting equations gives

$$P^* = \overline{P}, \; L^* = \overline{L}, S^* = \overline{S}, Q^* = \overline{Q},$$

where $(\overset{*}{P}, \overset{*}{L}, \overset{*}{S}, \overset{*}{Q})$ are values obtained for the system of ODE's. Thus the above finite difference discretization has the same fixed point as the system of original ODE's.

Let us now discuss the positivity. The discrete variable corresponding to Eq. (10) is [19,20]

$$P_k \geqslant 0, \; L_k \geqslant 0, \; S_k \geqslant 0, \; Q_k \geqslant 0 \Rightarrow P_{k+1} \geqslant 0, \; L_{k+1} \geqslant 0, \; S_{k+1} \geqslant 0, \; Q_{k+1} \geqslant 0. \tag{44}$$

Now making the transformation of variables

$$u_{k+1} = \sqrt{P_{k+1}},$$

in Eq. (39), we obtain quadratic equation for u_{k+1},

Table 1
Parameters used for numerical simulation.

Notation	Parameter definition	Value
μ	Natural death rate	0.04
γ	Recovery rate of infected class	0.03
β	Transmission coefficient	0.05
δ	Quit rate of smoking	0.03
d	Death rate for each class related to smoking disease	0.03
λ	Constant birth rate for the potential smokers individual	1.00

$$(1+\phi(\mu+d))u_{k+1}^2+(\beta\phi\sqrt{L_k})u_{k+1}-(P_k+\phi\lambda)=0.$$

Since our goal is to calculate, P_{k+1} from knowledge of $(\lambda,\mu,\beta,P_k,L_k)$ only the P_{k+1} variable is required under the transformation equation. Solution of the above quadratic equation is given by

$$u_{k+1}=\left[\frac{1}{2(1+\phi(\mu+d))}\right]\left[-(\beta\phi\sqrt{L_k})+\sqrt{(\beta\phi\sqrt{L_k})^2+4(1+\phi(\mu+d))(P_k+\phi\lambda)}\right]. \quad (45)$$

Similarly, the remaining equations of system (40)–(43) can be solved for variable at $(k+1)th$ time step:

$$L_{k+1}=\left[\frac{1}{(1+\phi(\gamma+\mu+d))}\right]\left((\beta\phi\sqrt{L_k})u_{k+1}+L_k\right), \quad (46)$$

$$S_{k+1}=\left[\frac{1}{(1+\phi(\delta+\mu+d))}\right](\gamma\phi L_{k+1}+S_k), \quad (47)$$

$$Q_{k+1}=\left[\frac{1}{(1+\phi(\mu+d))}\right](\delta\phi S_{k+1}+Q_k), \quad (48)$$

$$N_{k+1}=\left[\frac{1}{(1+\phi(\mu+d))}\right](\lambda\phi+N_k). \quad (49)$$

For $P_k\geqslant 0,\ L_k\geqslant 0,\ S_k\geqslant 0$, and $Q_k\geqslant 0$, Eq. (45) gives $u_{k+1}\geqslant 0 \Rightarrow P_{k+1}\geqslant 0$. By using these values in Eq. (46) we get $L_{k+1}\geqslant 0$. Likewise, proceed onward we obtain $S_{k+1}\geqslant 0,\ Q_{k+1}\geqslant 0$, and $N_{k+1}\geqslant 0$.

By all these values we conclude that all ODE's satisfying the positivity condition.

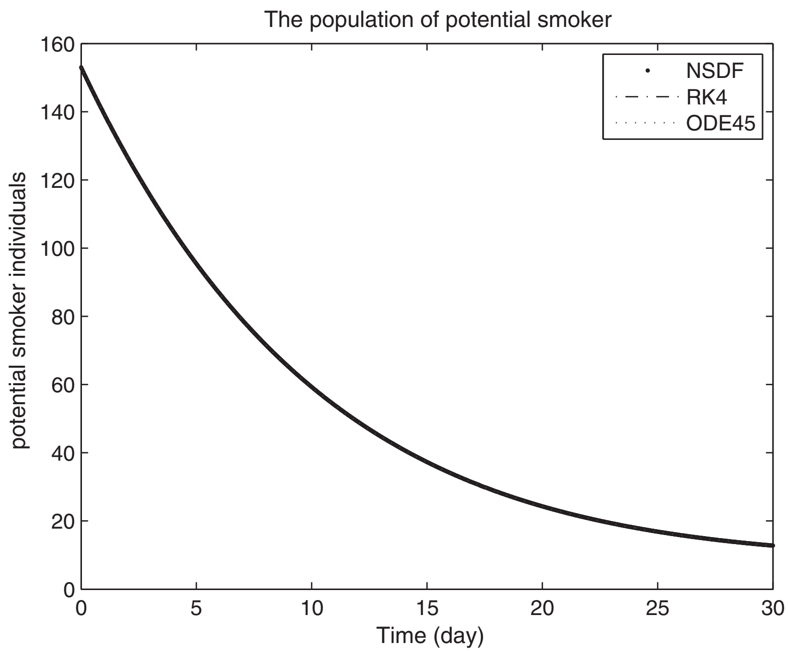

Fig. 1. The plot shows the potential smokers individuals.

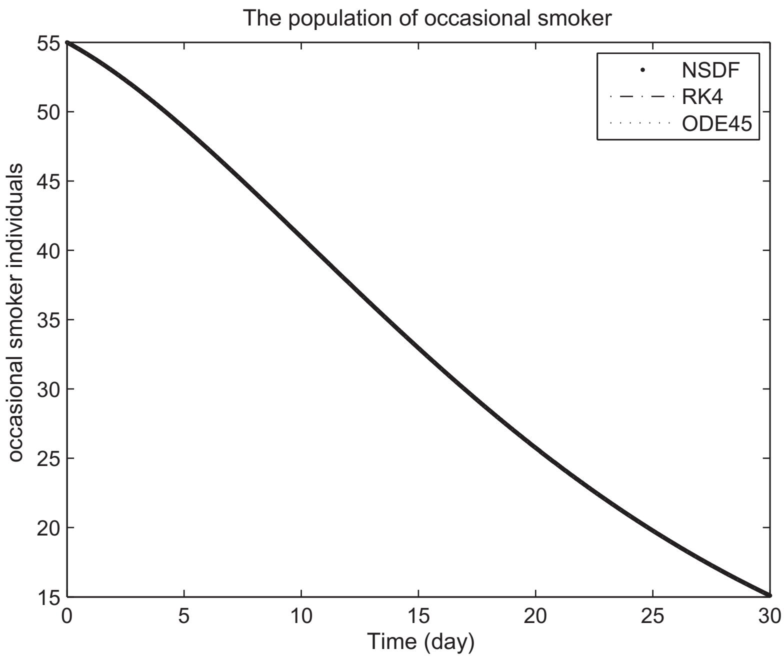

Fig. 2. The plot shows the occasional smoker individuals.

4. Numerical method and simulation

The NSFD algorithm is coded in computer algebra package Maple and we employ the Maple's built-in fourth-order Runge–Kutta procedure (RK4) and ODE45 [8]. The Maple environment variable `Digits` controlling the number of significant digits is set to 16 in all the calculations done in this paper. For the comparison, we use the set of parameters given in Table 1. To demonstrate the effectiveness of proposed algorithm as an approximate tool for solving the nonlinear system of differential Eqs. (11)–(15) for large time t, we apply this algorithm on the interval [0–30].

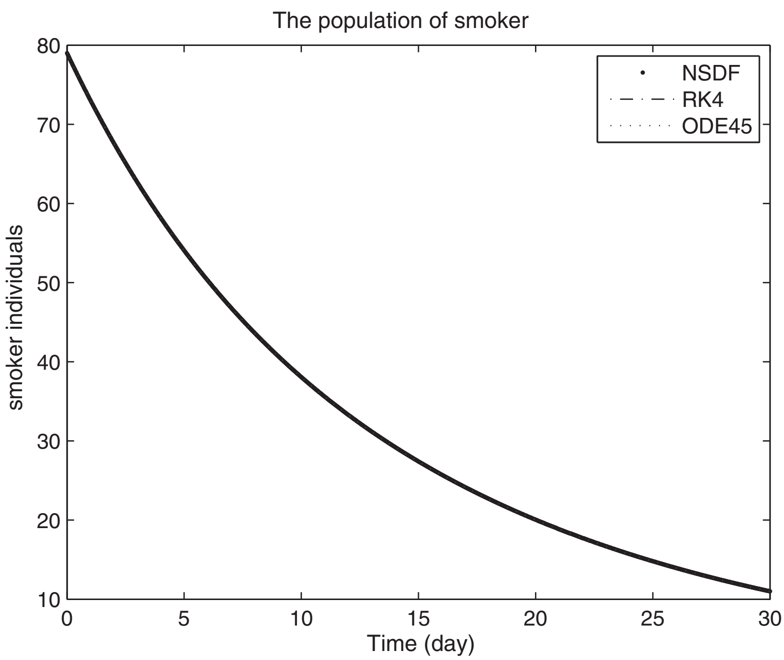

Fig. 3. The plot shows the smoker individuals.

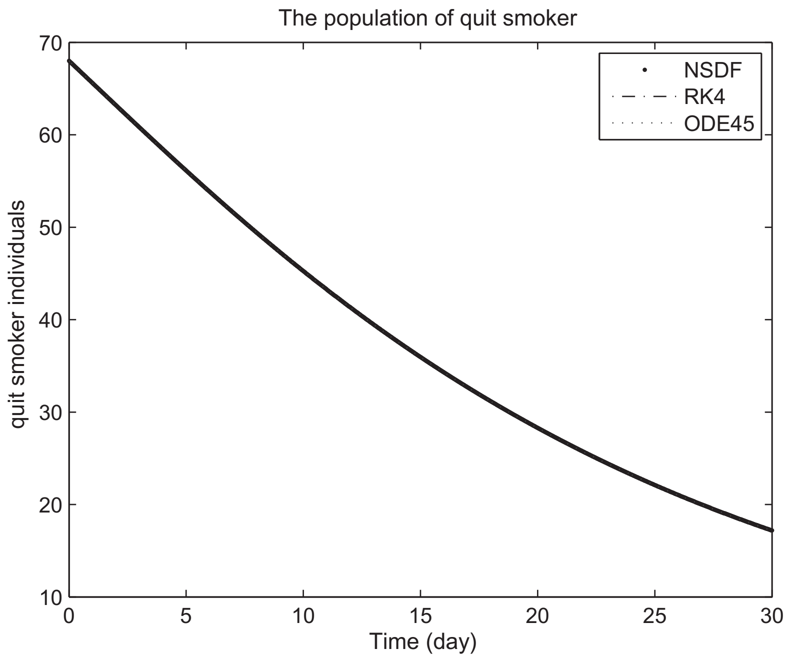

Fig. 4. The plot shows quit smoker individuals.

The approximate solutions of potential smokers $P(t)$, smokers $S(t)$, and quit smokers $Q(t)$ and total number of individuals $N(t)$ are represented in Figs. 1–5, respectively. From the graphical results represented in these figures, it can be seen that the results obtained using the non-standard finite difference method (NSFD) match the results of the classical Runge–Kutta method and ODE45 very well, which implies that the presented method can predict the behavior of these variables accurately for the region under consideration. Our numerical simulation shown that the smoker population of decreases sharply in the first few days and then decreases slowly. In this work we use one set of parameter for the equilibria of the dynamical

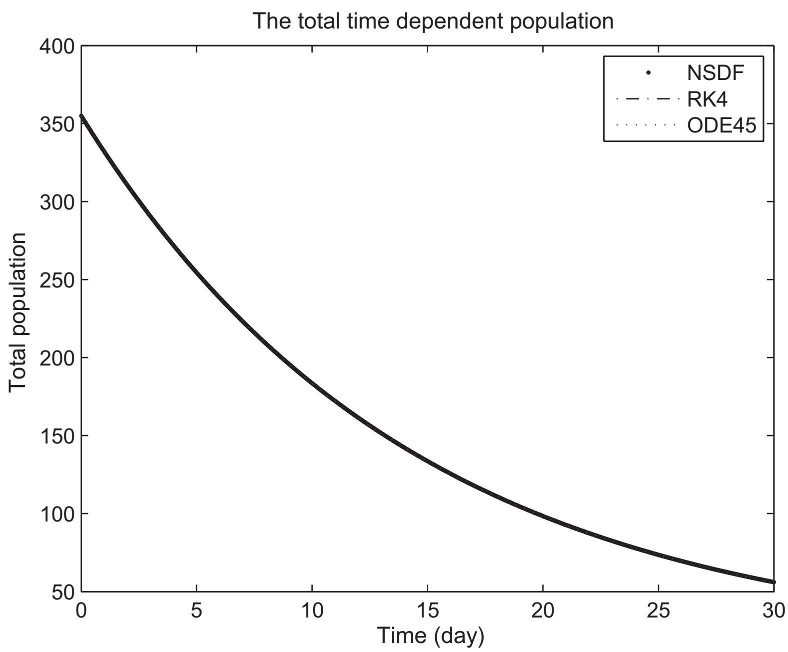

Fig. 5. The plot shows the total time dependent population.

system (11)–(15). Simulations with different sets of parameter values can be used in future to obtain sampling of possible behaviors of dynamical system.

5. Conclusion

In this paper, we presented giving-up smoking model for which interaction term is the square root of potential and occasional smokers of giving-up smoking model. Both bilinear incidence and saturated incidence rate in different models have been used for the dynamical behaviorial of different individuals. However, both these incidence rates did not help us to understand the possibility of having population go to extinction in a finite time. Therefore, we introduced the incidence rate in the form of square root to know the situation when the population go to extinction in a finite time.

First, we represented formulation of the model then we discussed the stability of the model. We showed that if the potential smoker did not reproduce, then the system was globally asymptotically stable. We also presented general solutions of model and constructed a discrete-time, finite difference scheme using non-standard finite difference (NSFD) method. The approximate solutions obtained by NSFD are highly accurate and valid for a long time. The reliability of the method and the reduction in size of computational domain give this method wider applicability.

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