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Solutions of non-linear oscillators by the modified differential transform method

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Abstract

A numerical method for solving nonlinear oscillators is proposed. The proposed scheme is based on the differential transform method (DTM), Laplace transform and Padé approximants. The modified differential transform method (MDTM) technique introduces an alternative framework designed to overcome the difficulty of capturing the periodic behavior of the solution, which is characteristic of oscillator equations, and give a good approximation to the true solution in a very large region. The numerical results demonstrate the validity and applicability of the new technique and a comparison is made with existing results.

Keywords: Non-linear oscillator; Differential transform method; Laplace transform; Padé approximants; Approximate analytical solution

1. Introduction

The most common methods for constructing approximate analytical solutions to the nonlinear oscillator equation are the perturbation methods. These methods include the harmonic balance method (HB) [1], the elliptic Lindstedt–Poincare method (LP) [1–3], the Krylov–Bogoliubov–Mitropolsky method (KBM) [4,5] the averaging [1, 6] and multiple scales method (MSM) [2] and are widely used to obtain approximate solutions of nonlinear oscillators. A feature common to all these methods is that they solve weakly nonlinear systems by using perturbation techniques to reduce these systems into simpler equations. Such a procedure changes the actual problem to make it tractable by conventional methods. In short, the physical problem is transformed into a purely mathematical one for which a solution is readily available. Recently, He [7,8] has proposed a new perturbation technique to eliminate the "small parameter" assumption. This method is called the homotopy perturbation method (HPM) and is applied to various nonlinear oscillator equations; for more details, see [9–13] and the references therein.

Our concern in this work is the derivation of approximate analytical oscillatory solutions for the non-linear oscillator equation

$$y''(t) + cy(t) = \varepsilon f(y(t), y'(t)), \qquad y(0) = a, \qquad y'(0) = b, \tag{1.1}$$

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where c is a positive real number and ε is a parameter (not necessarily small). We assume that the function $f(y(t), y'(t))$ is an arbitrary nonlinear function of its arguments. The modified differential transform method (DTM) will be employed in a straightforward manner without any need of linearization or smallness assumptions.

DTM was first applied in the engineering domain by [14]. DTM provides an efficient explicit and numerical solution with high accuracy, minimal calculations, avoidance of physically unrealistic assumptions. However, DTM has some drawbacks. By using DTM, we obtain a series solution, in practice a truncated series solution. This series solution does not exhibit the periodic behavior which is characteristic of oscillator equations and gives a good approximation to the true solution in a very small region.

In order to improve the accuracy of DTM, we use an alternative technique which modifies the series solution for non-linear oscillatory systems as follows: we first apply the Laplace transformation to the truncated series obtained by DTM, then convert the transformed series into a meromorphic function by forming its Padé approximants, and finally adopt an inverse Laplace transform to obtain an analytic solution, which may be periodic or a better approximation solution than the DTM truncated series solution. The first connection between series solution methods such as an Adomian decomposition method and Padé approximants was established in [15,16]. Finally, a numerical comparison has been made between the present method and the fourth-order Runge–Kutta method.

This paper is organized as follows: In Section 2, we describe the differential transform method and give a brief discussion of Padé approximants. In Section 3, the method is applied to a variety of examples to show the efficiency and simplicity of the method. At the end of the paper, there is a summary of the main conclusions.

2. Differential transform method

The differential transform method is a semi-numerical-analytic-technique that formulizes the Taylor series in a totally different manner. With this technique, the given differential equation and related boundary conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful for obtaining exact and approximate solutions of linear and nonlinear differential equations. There is no need for linearization or perturbations; large computational work and round-off errors are avoided. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations. The method is well addressed in [17–26]. The basic definitions of differential transformation are introduced as follows:

Definition 2.1. If $f(t)$ is analytic in the time domain T, then it will be differentiated continiously with respect to time t

$$\varphi(t,k) = \frac{\mathrm{d}^k f(t)}{\mathrm{d}t^k}, \quad \forall t \in T. \tag{2.1}$$

For $t = t_i$, then $\varphi(t,k) = \varphi(t_i,k)$, where k belongs to the set of non-negative integers, denoted as the K-domain. Therefore, Eq. (2.1) can be rewritten as

$$F(k) = \varphi(t_i,k) = \left[\frac{\mathrm{d}^k f(t)}{\mathrm{d}t^k}\right]\Bigg|_{t=t_i}, \quad \forall k \in K, \tag{2.2}$$

where $F(k)$ is called the spectrum of $f(t)$ at $t = t_i$ in the K-domain.

Definition 2.2. If $f(t)$ can be represented by the Taylor series, then it can be represented as

$$f(t) = \sum_{k=0}^{\infty}[(t-t_i)^k/k!]F(k). \tag{2.3}$$

Eq. (2.3) is called the inverse transform of $F(k)$. With the symbol D denoting the differential transform process, and upon combining Eqs. (2.2) and (2.3), we obtain

$$f(t) = \sum_{k=0}^{\infty}[(t-t_i)^k/k!]F(k) \equiv D^{-1}F(k).$$

Table 1
The fundamental operations of the differential transform method

Time function	Transformed function
$w(t) = \alpha u(t) \pm \beta v(t)$	$W(k) = \alpha U(k) \pm \beta V(k)$
$w(t) = \mathrm{d}^m u(t)/\mathrm{d}t^m$	$W(k) = \frac{(k+m)!}{k!} U(k+m)$
$w(t) = u(t)v(t)$	$W(k) = \sum_{l=0}^{k} U(l)V(k-l)$
$w(t) = t^m$	$W(k) = \delta(k-m) = \begin{cases} 1, & \text{if } k = m, \\ 0, & \text{if} k \neq m. \end{cases}$
$w(t) = \exp(t)$	$W(k) = 1/k!$
$w(t) = \sin(\omega t + \alpha)$	$W(k) = (\omega^k/k!)\sin(k\pi/2 + \alpha)$
$w(t) = \cos(\omega t + \alpha)$	$W(k) = (\omega^k/k!)\cos(k\pi/2 + \alpha)$

Using the differential transform, a differential equation in the domain of interest can be transformed into an algebraic equation in the K-domain and $f(t)$ can be obtained by the finite-term Taylor series expansion plus a remainder, as

$$f(t) = \sum_{k=0}^{N} [(t - t_i)^k / k!] F(k) + R_{N+1}(t).$$

The fundamental mathematical operations performed by DTM are listed in Table 1.

In addition to the above operations, the following theorem that can be deduced from Eqs. (2.2) and (2.3) is given below:

Theorem. *If* $f(x) = g_1(x) g_2(x) \ldots g_{m-1}(x) g_m(x)$, *then*

$$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1) G_2(k_2 - k_1) \ldots G_{n-1}(k_{n-1} - k_{n-2}) G_n(k - k_{n-1}). \tag{2.4}$$

The series solution (2.3) does not exhibit the periodic behavior that is characteristic of oscillator equations. It converges rapidly only in a small region; in the wide region, they may have very slow convergence rates, and then their truncations yield inaccurate results. Therefore, we follow the same technique proposed by [15,16], which modifies the series solution for non-linear oscillatory systems. In the modified DTM, we apply a Laplace transform to the series obtained by DTM, then convert the transformed series into a meromorphic function by forming its Padé approximants, and then invert the approximant to obtain an analytic solution, which may be periodic or a better approximation solution than the DTM truncated series solution.

2.1. Padé approximants

A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $y(x)$. The $[L/M]$ Padé approximants to a function $y(x)$ are given by [27].

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)}, \tag{2.5}$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The formal power series

$$y(x) = \sum_{i=1}^{\infty} a_i x^i, \tag{2.6}$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}), \tag{2.7}$$

determine the coefficients of $P_L(x)$ and $Q_M(x)$ by the equation.

Since we can clearly multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged, we impose the normalization condition

$$Q_M(0) = 1.0. \tag{2.8}$$

Finally we require that $P_L(x)$ and $Q_M(x)$ have no common factors.

If we write the coefficient of $P_L(x)$ and $Q_M(x)$ as

$$\left.\begin{array}{l} P_L(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L, \\ Q_M(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_M x^M \end{array}\right\}. \tag{2.9}$$

then, by (2.8) and (2.9), we may multiply (2.7) by $Q_M(x)$, which linearizes the coefficient equations. We can write out (2.7) in more detail as

$$\left.\begin{array}{l} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0, \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0, \\ \vdots \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0, \end{array}\right\} \tag{2.10}$$

$$\left.\begin{array}{l} a_0 = p_0, \\ a_0 + a_0 q_1 = p_1, \\ a_2 + a_1 q_1 + a_0 q_2 = p_2, \\ \vdots \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L \end{array}\right\}. \tag{2.11}$$

To solve these equations, we start with Eq. (2.10), which is a set of linear equations for all the unknown q's. Once the q's are known, then Eq. (2.11) gives an explicit formula for the unknown p's, which complete the solution.

If Eqs. (2.10) and (2.11) are nonsingular, then we can solve them directly and obtain Eq. (2.12) [27], where Eq. (2.12) holds, and if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right] = \frac{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} x^j & \sum_{j=M-1}^{L} a_{j-M+1} x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{bmatrix}}. \tag{2.12}$$

To obtain diagonal Padé approximants of different order Such as [2/2], [4/4] or [6/6] we can use the symbolic calculus software, Mathematica.

3. Applications

In order to assess the advantages and the accuracy of the modified DTM for solving non-linear oscillatory systems, we have applied the method to a variety of initial-value problems arising in non-linear dynamics. All the results are calculated by using Mathematica.

Example 3.1. Consider the following van der Pol equation

$$y''(t) + y(t) = \varepsilon[1 - y^2(t)]y'(t), \qquad y(0) = 0, \; y'(0) = 2, \tag{3.1}$$

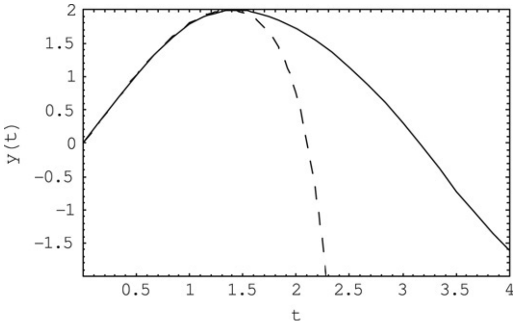

Fig. 1. Plots of displacement y versus time t. Runge–Kutta method (——); Eq. (3.5) (– – –).

subject to the initial conditions

$$y(0) = 0, \qquad y'(0) = 2. \tag{3.2}$$

Taking the differential transform of both sides of Eq. (3.1), we obtain the following recurrence relation:

$$Y(k+2) = \frac{1}{(k+1)(k+2)} \times \left[\varepsilon\left((k+1)Y(k+1) - \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}(k-k_2+1)Y(k_1)Y(k_2-k_1)Y(k-k_2+1)\right) - Y(k)\right]. \tag{3.3}$$

The initial conditions given in Eq. (3.2) can be transformed at $t_0 = 0$ as

$$Y(0) = 0, \qquad Y(1) = 2. \tag{3.4}$$

By using Eqs. (3.3) and (3.4) and (2.3), the following series solution is obtained:

$$y(t) = 2\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right) + \varepsilon\left(t^2 - \frac{5t^4}{6} + \frac{91t^6}{360} - \frac{41t^8}{1008}\right) + \cdots. \tag{3.5}$$

This series solution does not exhibit the periodic behavior that is characteristic of the oscillatory system (3.1) and (3.2). Comparison of the approximate solution (3.5) for $\varepsilon = 0.3$ and the solution obtained by the fourth-order Runge–Kutta method in Fig. 1 shows that it converges in a small region but yields a wrong solution in a wider region.

In order to improve the accuracy of the differential transform solution (3.5), we implement the modified DTM as follows:

Applying the Laplace transform to the series solution (3.5), yields

$$L[y(t)] = 2\left(\frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8}\right) + \varepsilon\left(\frac{2}{s^3} - \frac{20}{s^5} + \frac{182}{s^7} - \frac{1640}{s^9}\right) + \cdots. \tag{3.6}$$

For simplicity, let $s = 1/t$; then

$$L[y(t)] = 2(t^2 - t^4 + t^6 - t^8) + \varepsilon(2t^3 - 20t^5 + 182t^7 - 1640t^9) + \cdots. \tag{3.7}$$

The [4/4] Padé approximate for the terms containing $\varepsilon^0, \varepsilon^1, \ldots$ separately gives

$$\left[\frac{4}{4}\right] = 2\left(\frac{t^2}{1+t^2}\right) + \varepsilon\left(\frac{2t^3}{1+10t^2+9t^4}\right).$$

Recalling $t = 1/s$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right] = 2\left(\frac{1}{s^2+1}\right) + \varepsilon\left(\frac{2s}{s^4+10s^2+9}\right).$$

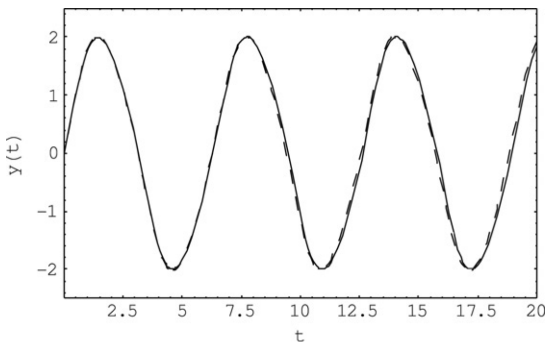

Fig. 2. Plots of displacement y versus time t. Runge–Kutta method (——); Eq. (3.8) (– – –).

By using the inverse Laplace transform to the [4/4] Padé approximant, we obtain the modified approximate solution

$$y(t) = 2\sin(t) + \varepsilon \cos(t)\sin^2(t). \tag{3.8}$$

Comparison of the modified approximate solution (3.8) and the solution obtained by the fourth-order Runge–Kutta method in Fig. 2 shows that the modified DTM greatly improves the differential transform truncated series (3.5) in the convergence rate and the accuracy.

Example 3.2. Consider the following non-linear equation

$$y''(t) + y(t) = -\varepsilon\, y^2(t)y'(t), \tag{3.9}$$

subject to the initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{3.10}$$

This equation can be appropriately called the "unplugged" van der Pol equation and all of its solutions are expected to oscillate with decreasing amplitude to zero. Momani [16] derived a numerical solution for the above equation using the modified decomposition method when $\varepsilon = 0.1$. For comparison with the solution obtained in Ref. [16] we set the parameter $\varepsilon = 0.1$ in this example.

Taking the differential transform of Eq. (3.9), the following recurrence relation is obtained:

$$Y(k+2) = -\frac{1}{(k+1)(k+2)}\left[\left(\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} 0.1(k-k_2+1)Y(k_1)Y(k_2-k_1)Y(k-k_2+1) + Y(k)\right)\right] \tag{3.11}$$

and taking $t_0 = 0$, the initial conditions given in Eq. (3.10) can be transformed to give

$$Y(0) = 1, \qquad Y(1) = 0. \tag{3.12}$$

By taking Eqs. (3.11) and (3.12), for $N = 7$, the following series solution is obtained:

$$\begin{aligned} y(t) &= 1 - 0.5t^2 + 0.0166667t^3 + 0.04125t^4 - 0.00665833t^5 \\ &\quad - 0.00098625t^6 + 0.00135318t^7 + O(t^8). \end{aligned} \tag{3.13}$$

Because Eqs. (3.9) and (3.10) form an oscillatory system, we apply the Laplace transform to the series solution (3.13), which yields

$$L[y(t)] = \frac{1}{s} - \frac{1}{s^3} + \frac{0.1}{s^4} + \frac{0.99}{s^5} - \frac{0.799}{s^6} - \frac{0.7101}{s^7} + \frac{6.82001}{s^8} - \cdots. \tag{3.14}$$

For simplicity, let $s = 1/t$; then

$$L[y(t)] = t - t^3 + 0.1t^4 + 0.99t^5 - 0.799t^6 - 0.7101t^7 + 6.82001t^8 - \cdots. \tag{3.15}$$

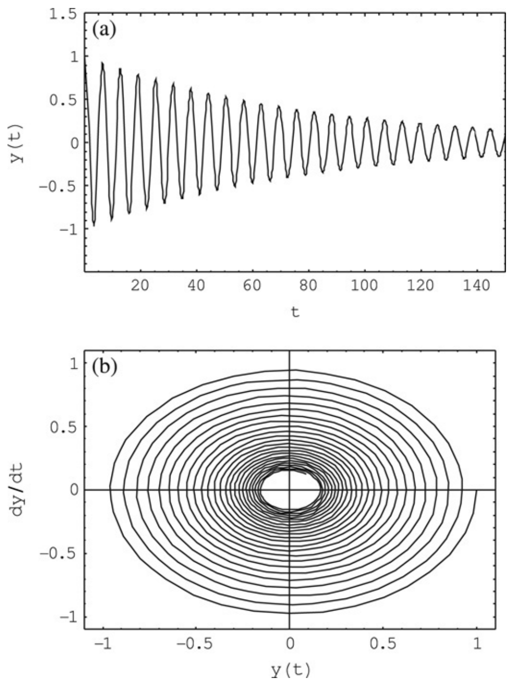

Fig. 3. Plots of Eq. (3.16): (a) displacement y versus time t and (b) phase plane diagram.

The [4/4] Padé approximant gives

$$\left[\frac{4}{4}\right] = \frac{t + 0.333333t^2 + 9.02778t^3 + 0.300444t^4}{1 + 0.333333t + 10.0278t^2 + 0.533778t^3 + 9.00444t^4}.$$

Recalling $t = 1/s$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right] = \frac{s^3 + 0.333333s^2 + 9.02778s + 0.300444}{s^4 + 0.333333s^3 + 10.0278s^2 + 0.533778s + 9.00444}.$$

By using the inverse Laplace transform to the [4/4] Padé approximant, we obtain the modified solution

$$\begin{aligned} y(t) &= a_1 e^{(-0.154136-2.99971i)t} + a_2 e^{(-0.154136+2.99971i)t} \\ &\quad + a_3 e^{(-0.0125308-0.998948i)t} + a_4 e^{(-0.0125308+0.998948i)t}, \end{aligned} \tag{3.16}$$

where

$$a_1 = 0.000284635 - 0.00152866i, \qquad a_2 = 0.000284635 + 0.00152866i,$$
$$a_3 = 0.499715 + 0.0109027i, \qquad a_4 = 0.499715 - 0.0109027i.$$

Fig. 3 shows the displacement and phase diagram of the modified approximate solution (3.16). The results of our computations are in good agreement with the results obtained by the numerical solution of Momani [9] using the modified decomposition method.

Example 3.3. Consider the following Duffing equation:

$$y''(t) + y(t) + 0.3y^3(t) = 0, \tag{3.17}$$

subject to the initial conditions

$$y(0) = 0, \qquad y'(0) = 1. \tag{3.18}$$

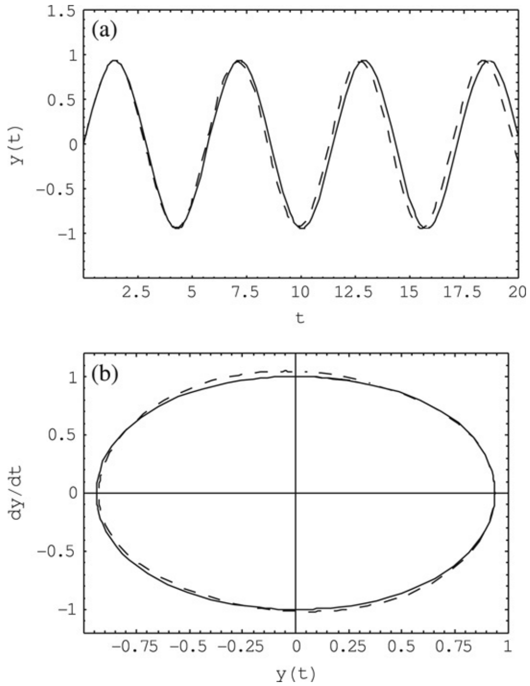

Fig. 4. Plots of (a) displacement y versus time t and (b) phase plane diagram. Runge–Kutta method (——); Eq. (3.23) (– – –).

Choosing $t_0 = 0$, Eqs. (3.17) and (3.18) are transformed as follows:

$$Y(k+2) = -\frac{1}{(k+1)(k+2)}\left(0.3\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} Y(k_1)Y(k_2-k_1)Y(k-k_2) + Y(k)\right). \tag{3.19}$$

$$Y(0) = 0, \qquad Y(1) = 1. \tag{3.20}$$

By taking Eqs. (3.19) and (3.20), for $N = 9$, the following series solution is obtained:

$$y(t) = t - 0.166667t^3 - 0.00666667t^5 + 0.00373016t^7 - 0.000315697t^9 + O(t^{11}). \tag{3.21}$$

Applying the Laplace transform to the series solution (3.21), yields

$$L[y(t)] = \frac{1}{s^2} - \frac{1}{s^4} - \frac{0.8}{s^6} + \frac{18.8}{s^8} - \frac{114.56}{s^{10}} - \cdots. \tag{3.22}$$

Setting $s = 1/t$ in Eq. (3.22) and calculating the [4/4] Padé approximant gives

$$\left[\frac{4}{4}\right] = \frac{9t^4 + t^2}{10.8t^4 + 10t^2 + 1}.$$

Recalling $t = 1/s$, and by using the inverse Laplace transformation to the [4/4] Padé approximant, we obtain the modified solution

$$y(t) = 0.928746\sin(1.10982t) - 0.0103828\sin(2.96113t). \tag{3.23}$$

The graphs of the displacement and phase diagram are sketched in Fig. 4 and are compared with the numerical solution of the Runge–Kutta method.

Example 3.4. Consider the following Helmholtz equation:

$$y''(t) + y(t) + 0.1y^2(t) = 0, \tag{3.24}$$

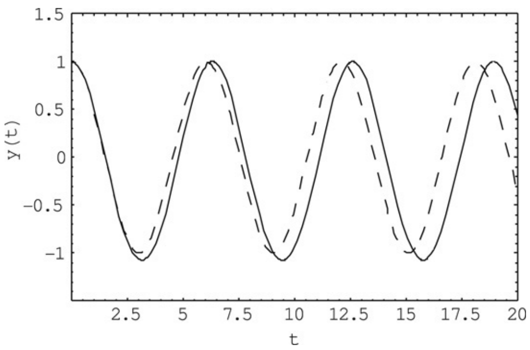

Fig. 5. Plots of (a) displacement y versus time t. Runge–Kutta method (——); Eq. (3.30) (– – –).

subject to the initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{3.25}$$

Taking the differential transform of Eq. (3.24), we have

$$Y(k+2) = -\frac{1}{(k+1)(k+2)}\left(0.1\sum_{l=0}^{k} Y(l)Y(k-l) + Y(k)\right). \tag{3.26}$$

Taking $t_0 = 0$, the initial conditions given in Eq. (3.25) can be transformed to give

$$Y(0) = 1, \qquad Y(1) = 0. \tag{3.27}$$

Utilizing the recurrence relation in Eq. (3.26) and the transformed initial conditions in Eq. (3.27), the following series solution up to $O(t^{10})$ is obtained:

$$y(t) = 1 - 0.55t^2 + 0.055t^4 - 0.00320833t^6 + 0.000176786t^8 + O(t^{10}). \tag{3.28}$$

Applying the Laplace transform to the series solution (3.28), yields

$$L[y(t)] = \frac{1}{s} - \frac{1}{s^3} + \frac{1.32}{s^5} - \frac{2.31}{s^7} + \frac{7.128}{s^9} + \cdots. \tag{3.29}$$

The [4/4] Padé approximant gives

$$\left[\frac{4}{4}\right] = \frac{t^3 + 6.7t}{t^4 + 7.8t^2 + 7.26}.$$

Recalling $t = 1/s$, and by using the inverse Laplace transformation to the [4/4] Padé approximant, we obtain the modified solution

$$y(t) = 0.996529\cos(1.03944t) + 0.00347117\cos(2.59221t). \tag{3.30}$$

The graph of the displacement is sketched in Fig. 5 and is compared with the numerical solution of the fourth-order Runge–Kutta method.

4. Conclusion

The modified DTM is an efficient method for calculating periodic solutions for non-linear oscillatory systems. All the examples show that the results of the present method are in excellent agreement with those obtained by the fourth-order Runge–Kutta method and modified Adomian decomposition method [16] even for moderately large values of the parameter ε. These examples indicate that the modified differential transform greatly improves DTM's truncated series solution in the convergence rate, and that it often yields the true analytic solution. The results also show that the method is a very promising one and might find wide applications.

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