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Solutions of a fractional oscillator by using differential transform method

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ABSTRACT

In this paper, we present an efficient algorithm for solving a fractional oscillator using the differential transform method. The fractional derivatives are described in the Caputo sense. The application of differential transform method, developed for differential equations of integer order, is extended to derive approximate analytical solutions of a fractional oscillator. The method provides the solution in the form of a rapidly convergent series. Numerical examples are used to illustrate the preciseness and effectiveness of the proposed method.

1. Introduction

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular [1–3].

In this paper, we will consider the dynamics of the so-called driven fractional oscillator. This fractional oscillator is obtained by replacing the second time derivative term in the corresponding harmonic oscillator by a fractional derivative of order α with $1 < \alpha \leq 2$. The derivatives are understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 2$, the fractional system of oscillators reduces to the standard system of simple harmonic oscillators. Some aspects of such a system have been studied previously by other researchers [4–6].

The differential transform method was first applied in the engineering domain by Zhou [7,8]. In general, the differential transform method is applied to the solution of electric circuit problems. The differential transform method is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the differential transform method obtains a polynomial series solution by means of an iterative procedure. Recently, the application of differential transform method is successfully extended to obtain analytical approximate solutions to linear and nonlinear ordinary differential equations of fractional order [9,10]. A comparison between the differential transform method and Adomian decomposition method for solving fractional differential equations is given in [9]. The fact that the differential transform method solves nonlinear equations without using Adomian polynomials can be considered as an advantage of this method over Adomian decomposition method [11–13]. The aim of our study is to extend the application of the differential transform method [9]

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to fractional oscillators. According to the authors knowledge, this paper represents the first application of the differential transform method to fractional oscillators.

There are several definitions of a fractional derivative of order $\alpha > 0$ [1,2]. The two most commonly used definitions are the Riemann–Liouville and Caputo. Each definition uses Riemann–Liouville fractional integration and derivatives of whole order. The difference between the two definitions is in the order of evaluation. Riemann–Liouville fractional integration of order α is defined as

$$J_{x_0}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad \alpha > 0,\ x > 0. \tag{1.1}$$

The next two equations define Riemann–Liouville and Caputo fractional derivatives of order α, respectively,

$$D_{x_0}^{\alpha} f(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[J^{m-\alpha} f(x)\right], \tag{1.2}$$

$$D_{*x_0}^{\alpha} f(x) = J^{m-\alpha}\left[\frac{\mathrm{d}^m}{\mathrm{d}x^m} f(x)\right], \tag{1.3}$$

where $m-1 < \alpha \le m$ and $m \in N$. For now, the Caputo fractional derivative will be denoted by D_*^{α} to maintain a clear distinction with the Riemann–Liouville fractional derivative. The Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. The Riemann–Liouville fractional derivative is computed in the reverse order. We have chosen to use the Caputo fractional derivative because it allows traditional initial and boundary conditions to be included in the formulation of the problem, but for homogeneous initial condition assumption, these two operators coincide. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann–Liouville and Caputo types see [1,2].

2. Fractional differential transform method

In this section, we introduce the fractional differential transform method used in this paper to obtain approximate analytical solutions for a fractional oscillator. This method has been developed by Arikoglu and Ozkol [9] as follows:

The fractional differentiation in Riemann–Liouville sense is defined by

$$D_{x_0}^{q} f(x) = \frac{1}{\Gamma(m-q)} \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\int_{x_0}^{x} \frac{f(t)}{(x-t)^{1+q-m}}\mathrm{d}t\right], \tag{2.1}$$

for $m-1 \le q < m$, $m \in Z^+$, $x > x_0$. Let us expand the analytical and continuous function $f(x)$ in terms of a fractional power series as follows:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^{\frac{k}{\alpha}}, \tag{2.2}$$

where α is the order of fraction and $F(k)$ is the fractional differential transform of $f(x)$.

In order to avoid fractional initial and boundary conditions, we define the fractional derivative in the Caputo sense. The relation between the Riemann–Liouville operator and Caputo operator is given by :

$$D_{*x_0}^{q} f(x) = D_{x_0}^{q}\left[f(x) - \sum_{k=0}^{m-1} \frac{1}{k!}(x-x_0)^k f^{(k)}(x_0)\right]. \tag{2.3}$$

Setting $f(x) = f(x) - \sum_{k=0}^{m-1} \frac{1}{k!}(x-x_0)^k f^{(k)}(x_0)$ in Eq. (2.1) and using Eq. (2.3), we obtain fractional derivative in the Caputo sense [2] as follows:

$$D_{*x_0}^{q} f(x) = \frac{1}{\Gamma(m-q)} \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left\{\int_{x_0}^{x}\left[\frac{f(t) - \sum\limits_{k=0}^{m-1} \frac{1}{k!}(t-x_0)^k f^{(k)}(x_0)}{(x-t)^{1+q-m}}\right]\mathrm{d}t\right\}. \tag{2.4}$$

Since the initial conditions are implemented to the integer order derivatives, the transformation of the initial conditions are defined as follows:

$$F(k) = \begin{cases} \dfrac{1}{(\frac{k}{\alpha})!}\left[\dfrac{\mathrm{d}^{\frac{k}{\alpha}} f(x)}{\mathrm{d}x^{\frac{k}{\alpha}}}\right]_{x=x_0}, & \text{if } \dfrac{k}{\alpha} \in Z^+, \quad \text{for } k = 0, 1, 2, \ldots, (q\alpha - 1), \\ 0, & \text{if } \dfrac{k}{\alpha} \notin Z^+, \end{cases} \tag{2.5}$$

where q is the order of fractional differential equation considered. The following theorems that can be deduced from Eqs. (2.1) and (2.2) are given below, for proofs and details see [9]:

Theorem 1. If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$.

Theorem 2. If $f(x) = g(x)h(x)$, then $F(k) = \sum_{l=0}^{k} G(l)H(k-l)$.

Theorem 3. If $f(x) = g_1(x)g_2(x)\dots g_{n-1}(x)g_n(x)$, then $F(k) = \sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)\dots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1})$.

Theorem 4. If $f(x) = (x-x_0)^p$, then $F(k) = \delta(k-\alpha p)$ where, $\delta(k) = \begin{cases} 1, & \text{if } k = 0,\\ 0, & \text{if } k \neq 0.\end{cases}$

Theorem 5. If $f(x) = D_{x_0}^{q}[g(x)]$, then $F(k) = \frac{\Gamma(q+1+\frac{k}{\alpha})}{\Gamma(1+\frac{k}{\alpha})}G(k+\alpha q)$.

Theorem 6. For the production of fractional derivatives in the most general form, if $f(x) = \frac{d^{q_1}}{dx^{q_1}}[g_1(x)]\frac{d^{q_2}}{dx^{q_2}}[g_2(x)]\cdots\frac{d^{q_{n-1}}}{dx^{q_{n-1}}}[g_{n-1}(x)]\frac{d^{q_n}}{dx^{q_n}}[g_n(x)]$, then $F(k) = \sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{\Gamma\left[q_1+1+\frac{k_1}{\alpha}\right]}{\Gamma\left[1+\frac{k_1}{\alpha}\right]}\frac{\Gamma\left[q_2+1+\frac{k_2-k_1}{\alpha}\right]}{\Gamma\left[1+\frac{k_2-k_1}{\alpha}\right]}\cdots\frac{\Gamma\left[q_{n-1}+1+\frac{k_{n-1}-k_{n-2}}{\alpha}\right]}{\Gamma\left[1+\frac{k_{n-1}-k_{n-2}}{\alpha}\right]}$ $\times\frac{\Gamma\left[q_n+1+\frac{k-k_{n-1}}{\alpha}\right]}{\Gamma\left[1+\frac{k-k_{n-1}}{\alpha}\right]}G_1(k_1+\alpha q_1)G_2(k_2-k_1+\alpha q_2)\cdots G_{n-1}(k_{n-1}-k_{n-2}+\alpha q_{n-1})G_n(k-k_{n-1}+\alpha q_n)$, where $\alpha q_i \in Z^+$ for $i = 1, 2, 3, \dots, n$.

3. Numerical examples

Example 1. *Consider the following fractional differential equation* [4]*:*

$$\frac{d^\alpha u}{dt^\alpha} + \omega^\alpha u(t) = f(t), \quad 1 < \alpha \leq 2, \tag{3.1}$$

subject to the initial conditions

$$u(0) = 1, \qquad u'(0) = 0. \tag{3.2}$$

This equation describes a simple harmonic fractional oscillator where the forcing function in this case is $f(t) = 0$. Eq. (3.1) is transformed by using Theorems 1 and 5 as follows:

$$U(k+\beta\alpha) = \frac{-\omega^\alpha\Gamma(1+\frac{k}{\beta})U(k)}{\Gamma(\alpha+1+\frac{k}{\beta})}, \tag{3.3}$$

where β is the unknown value of the fraction of α. Initial conditions in Eq. (3.2) are transformed by using Eq. (2.5) as follows:

$$U(0) = 1, \qquad U(k) = 0 \quad \text{for } k = 1, 2, \dots, \beta\alpha - 1. \tag{3.4}$$

Using Eqs. (3.3) and (3.4), $U(k)$ for $k = \beta\alpha, \beta\alpha+1, \beta\alpha+2, \dots,$ is calculated, and using the inverse transformation rule in Eq. (2.2), $u(t)$ is calculated for different values of α. For $\alpha = 2$, the first few terms of the series solution are given by

$$u(t) = 1 - \frac{(\omega t)^2}{2} + \frac{(\omega t)^4}{24} - \frac{(\omega t)^6}{720} + \frac{(\omega t)^8}{40320} - \cdots. \tag{3.5}$$

Eq. (3.5) is the solution of a simple harmonic oscillator and given by $u(t) = \cos(\omega t)$. By taking $\alpha = 1.7$, $u(t)$ is obtained as follows:

$$u(t) = 1 - \frac{(\omega t)^{\frac{17}{10}}}{\Gamma(\frac{27}{10})} + \frac{(\omega t)^{\frac{17}{5}}}{\Gamma(\frac{22}{5})} - \frac{(\omega t)^{\frac{51}{10}}}{\Gamma(\frac{61}{10})} + \frac{(\omega t)^{\frac{34}{5}}}{\Gamma(\frac{39}{10})} - \cdots. \tag{3.6}$$

Finally, the following series solution is obtained for $\alpha = 1.9$:

$$u(t) = 1 - \frac{(\omega t)^{\frac{19}{10}}}{\Gamma(\frac{29}{10})} + \frac{(\omega t)^{\frac{19}{5}}}{\Gamma(\frac{24}{5})} - \frac{(\omega t)^{\frac{57}{10}}}{\Gamma(\frac{67}{10})} + \frac{(\omega t)^{\frac{38}{5}}}{\Gamma(\frac{43}{10})} - \cdots. \tag{3.7}$$

Fig. 1 shows the approximate solution for Eq. (3.1) obtained for $\alpha = 1.7$, 1.9 and 2. It should be noted that the 460 terms were used in evaluating the approximate solutions for Fig. 1. Comparison between these results shows how the displacement of the fractional oscillator varies as a function of time and how this time variation depends on the parameter α. It can be seen that the behavior of the driven fractional oscillator is very similar to the behavior of the damped harmonic oscillator, where the motion is still oscillatory, but the total energy decrease and the phase plane diagram is no longer a closed curve, but a logarithmic spiral.

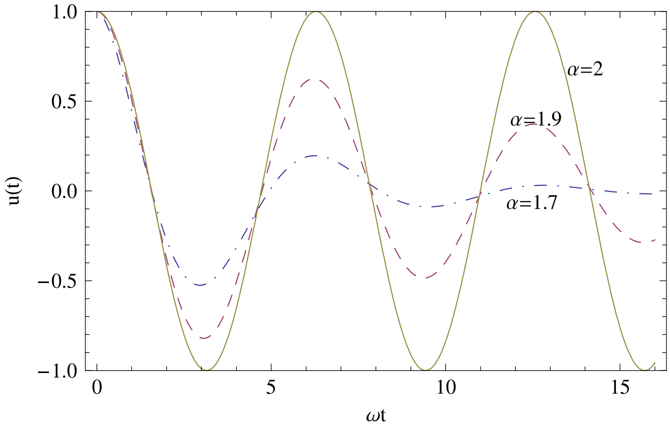

Fig. 1. $u(t)$ function of Eq. (3.1) for different values of α.

Example 2. *Now, we consider the driven fractional oscillator system of the form* [4]*:*

$$\frac{d^\alpha u}{dt^\alpha} + \omega^\alpha u(t) = f(t), \quad 1 < \alpha \leq 2, \tag{3.8}$$

subject to the initial conditions

$$u(0) = c, \qquad u'(0) = 0, \tag{3.9}$$

where the forcing function in this case is the step function given by

$$f(t) = \begin{cases} A, & \text{for } t > 0, A \text{ constant;} \\ 0, & \text{for } t < 0. \end{cases} \tag{3.10}$$

Using Theorems 1, 4 and 5, Eq. (3.8) can be transformed as follows:

$$U(k + \beta\alpha) = \frac{\Gamma(1 + \frac{k}{\beta})[A\delta(k) - \omega^\alpha U(k)]}{\Gamma(\alpha + 1 + \frac{k}{\beta})}. \tag{3.11}$$

The conditions in Eq. (3.9) can be transformed by using Eq. (2.5) as

$$U(0) = c, \qquad U(k) = 0 \quad \text{for } k = 1, 2, \ldots, \beta\alpha - 1. \tag{3.12}$$

Using Eqs. (3.11) and (3.12), $U(k)$ can easily be calculated for $k = \beta\alpha, \beta\alpha + 1, \beta\alpha + 2, \ldots$. Then by using the inverse transformation rule in Eq. (2.2), $u(t)$ is obtained as follows:

$$u(t) = c + t^{\frac{17}{10}} \left[\frac{A}{\Gamma(\frac{27}{10})} - \frac{c\omega^{\frac{17}{10}}}{\Gamma(\frac{27}{10})} \right] + t^{\frac{17}{5}} \left[-\frac{A\omega^{\frac{17}{10}}}{\Gamma(\frac{22}{5})} + \frac{c\omega^{\frac{17}{5}}}{\Gamma(\frac{22}{5})} \right] + \cdots. \tag{3.13}$$

It is to be noted that the 430 terms were used in evaluating the approximate solutions for Fig. 2. To examine the effect of the step function $f(t) = A$, we choose $u(0) = c = 0$ and $A = 1$. Fig. 2 shows the response function for $\alpha = 1.7$, 1.9 and 2. The results here are very similar to the results of the previous example. The behavior of the driven fractional oscillator for the step function is similar to the behavior of the damped oscillator.

Example 3. *In this example, we choose the forcing function to be the sinusoidal function* $f(t) = \sin(\omega t)$*. That is, we consider the following equation* [4]*:*

$$\frac{d^\alpha u}{dt^\alpha} + \omega^\alpha u(t) = \sin(\omega t), \quad 1 < \alpha \leq 2, \tag{3.14}$$

subject to the initial conditions

$$u(0) = 0, \qquad u'(0) = 0. \tag{3.15}$$

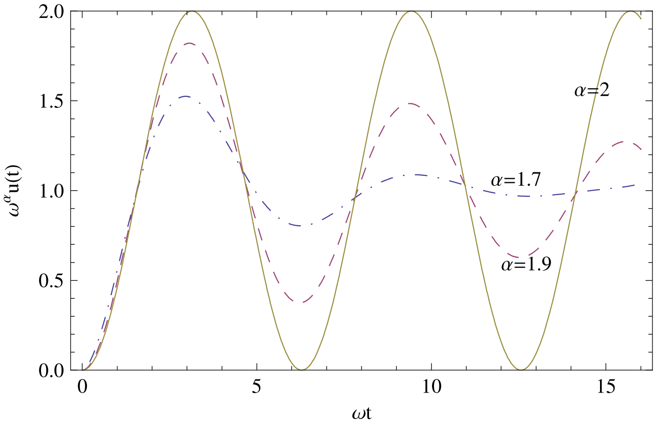

Fig. 2. $\omega^{\alpha}u(t)$ function of Eq. (3.8) for different values of α.

Eq. (3.14) is transformed by using Theorems 1 and 5 as follows:

$$U(k+\beta\alpha)=\frac{\Gamma(1+\frac{k}{\beta})[S(k)-\omega^{\alpha}U(k)]}{\Gamma(\alpha+1+\frac{k}{\beta})},\tag{3.16}$$

where $S(k)$ is the fractional differential transform of $\sin(\omega t)$ that can be evaluated using Eq. (2.2) as

$$S(k)=\sum_{i=0}^{\infty}\frac{(-1)^{i}\omega^{2i+1}}{(2i+1)!}\delta[k-\beta(2i+1)].\tag{3.17}$$

The conditions in Eq. (3.15) can be transformed by using Eq. (2.5) as

$$U(k)=0\quad \text{for } k=0,1,2,\ldots,\beta\alpha-1.\tag{3.18}$$

Using Eqs. (3.16) and (3.17), $U(k)$ for $k=\alpha\beta,\alpha\beta+1,\ldots,$ is calculated and using the inverse transformation rule in Eq. (2.2), $u(t)$ is calculated for different values of α. For the values of $\alpha=1.75$ and $\beta=4$, $u(t)$ is obtained as follows:

$$u(t)=-\frac{32t^{\frac{9}{2}}\omega^{\frac{11}{4}}}{945\sqrt{\pi}}+\frac{128t^{\frac{13}{2}}\omega^{\frac{19}{4}}}{135\,135\sqrt{\pi}}-\frac{t^{8}\omega^{\frac{25}{4}}}{40\,320}-\frac{512t^{\frac{17}{2}}\omega^{\frac{27}{4}}}{34\,459\,425\sqrt{\pi}}+\cdots.\tag{3.19}$$

By taking $\alpha=1.9$ and $\beta=10$, $u(t)$ is obtained as follows:

$$u(t)=\frac{2048t^{\frac{21}{2}}\omega^{\frac{43}{15}}}{13\,749\,310\,575\sqrt{\pi}}-\frac{8192t^{\frac{25}{2}}\omega^{\frac{53}{5}}}{7\,905\,853\,580\,625\sqrt{\pi}}+\frac{32768t^{\frac{29}{2}}\omega^{\frac{63}{5}}}{6\,190\,283\,353\,629\,375\sqrt{\pi}}-\cdots.\tag{3.20}$$

Fig. 3 shows the response function for $\alpha=1.75$ and 1.9. It is to be noted that the 400 terms were used in evaluating the approximate solution for Fig. 3.

Example 4. *Finally, we consider nonlinear fractional Van Der Pol oscillator of the form* [6]*:*

$$\frac{d^{2}u}{dt^{2}}+\mu(u^{2}(t)-1)\frac{d^{\alpha}u}{dt^{\alpha}}+u(t)=a\sin(\omega t),\quad 1<\alpha\le 2,\tag{3.21}$$

subject to the initial conditions

$$u(0)=0,\qquad u'(0)=0.\tag{3.22}$$

Eq. (3.21) is transformed by using Theorems 1, 2 and 5 as follows:

$$U(k+2\beta)=\frac{\Gamma(1+\frac{k}{\beta})}{\Gamma(3+\frac{k}{\beta})}\left[aS(k)-U(k)-\mu\sum_{l=0}^{k}[\delta(l-2\beta)-\delta(l)]\frac{\Gamma(\alpha+1+\frac{k-1}{\beta})}{\Gamma(1+\frac{k-1}{\beta})}U(k-l+\alpha\beta)\right],\tag{3.23}$$

where $S(k)$ is the fractional differential transform of $\sin(\omega t)$ that can be evaluated using Eq. (2.2) as

$$S(k)=\sum_{i=0}^{\infty}\frac{(-1)^{i}\omega^{2i+1}}{(2i+1)!}\delta[k-\beta(2i+1)].\tag{3.24}$$

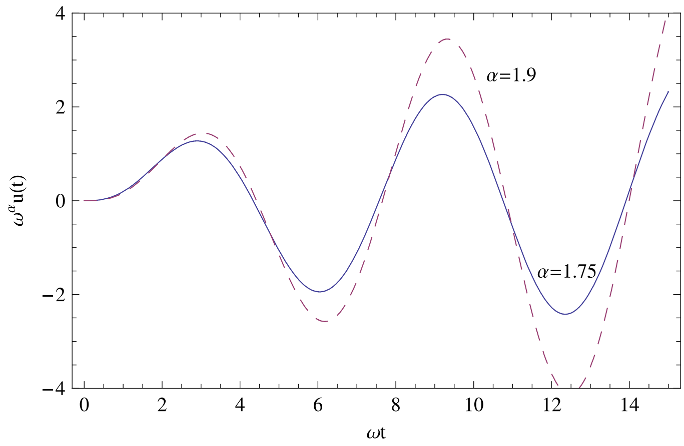

Fig. 3. $\omega^{\alpha}u(t)$ function of Eq. (3.14) for different values of α.

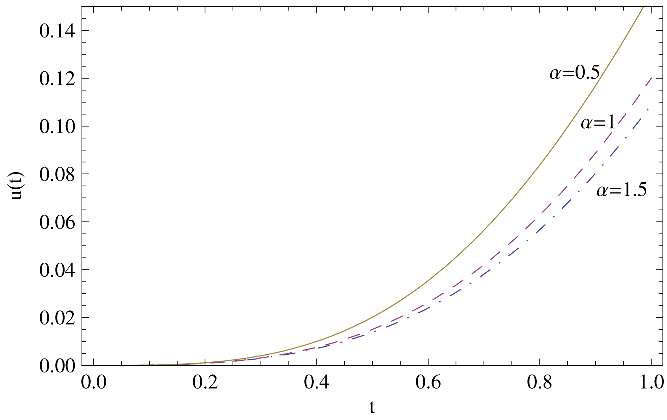

Fig. 4. $u(t)$ function of Eq. (3.21) for different values of α.

The conditions in Eq. (3.22) can be transformed by using Eq. (2.5) as

$$U(k) = 0 \quad \text{for } k = 0, 1, 2, \ldots, 2\beta\alpha - 1. \tag{3.25}$$

Using Eqs. (3.23) and (3.25), $U(k)$ for $k = 2\beta\alpha, 2\beta\alpha + 1, \ldots,$ is calculated, and using the inverse transformation rule in Eq. (2.2), $u(t)$ is calculated for different values of α. For the values of $\alpha = 0.5$ and $\beta = 2$, $u(t)$ is obtained as follows:

$$\begin{aligned} u(t) &= \frac{1}{6}at^3\omega + \frac{32at^{\frac{9}{2}}\mu\omega}{945\sqrt{\pi}} + \frac{1}{720}at^6\mu^2\omega + t^5\left[-\frac{a\omega}{120} - \frac{a\omega^3}{120}\right] \\ &\quad + t^{\frac{13}{2}}\left[-\frac{2272a\mu\omega}{135\,135\sqrt{\pi}} - \frac{128a\mu\omega^3}{135\,135\sqrt{\pi}}\right] + \cdots. \end{aligned} \tag{3.26}$$

By taking $\alpha = 1$, $u(t)$ is obtained as follows:

$$u(t) = \frac{1}{6}at^3\omega + \frac{1}{24}at^4\mu\omega + t^5\left[-\frac{a\omega}{120} + \frac{a\mu^2\omega}{120} + \frac{a\omega^3}{120}\right] + \cdots. \tag{3.27}$$

Finally, the following series solution is obtained for $\alpha = 1.5$:

$$u(t) = \frac{1}{6}at^3\omega + \frac{16at^{\frac{7}{2}}\mu\omega}{105\sqrt{\pi}} + \frac{1}{24}at^4\mu^2\omega + \frac{32at^{\frac{9}{2}}\mu^3\omega}{945\sqrt{\pi}} + t^5\left[\frac{a\omega}{120} + \frac{a\mu^4\omega}{120} - \frac{a\omega^3}{120}\right] + \cdots. \tag{3.28}$$

Fig. 4 shows the response function for $\alpha = 0.5$, 1 and 1.5. The parameters are taken as $\mu = 1.05$, $a = 1.20$ and $\omega = 0.5$. It is to be noted that the 55 terms were used in evaluating the approximate solutions for Fig. 4.

4. Conclusions

In this article, the application of differential transform method was extended to obtain explicit and numerical solutions of linear and nonlinear fractional oscillators. The method provides the solution as infinite series of functions with easily computable components.

The response functions of different force functions are obtained for different values of α ($1 < \alpha \leq 2$). The numerical results showed that the behavior of the fractional oscillator is similar to the behavior of the damped harmonic oscillator. It may be concluded that the displacement functions are able to describe processes intermediate between exponential decay ($\alpha = 1$) and pure sinusoidal oscillation ($\alpha = 2$).

Finally, the differential transform method is used to construct an approximate analytical solution for the nonlinear fractional Van Der Pol oscillator.

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