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On a fractional integral equation of periodic functions involving Weyl–Riesz operator in Banach algebras

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Abstract

In this paper, we study the existence of periodic solutions for a nonlinear integral equation of periodic functions involving Weyl–Riesz fractional integral operator under the mixed generalized *Lipschitz*, *Carathéodory* and monotonicity conditions. The fixed point theorems due to Dhage are the main tool in carrying out our proofs.

Keywords: Weyl–Riesz operators; Fractional integral equation; Periodic solutions; Hammerstein type

1. Introduction and preliminaries

Recently, the topic of nonlinear integral and differential equations in Banach algebras is received the attention of several authors [1–3]. Integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. On the other hand, fractional integral and differential equations have been shown by many researchers to adequately describe the operation of a variety of physical and biological processes and systems. Examples include viscoelasticity, electroanalytical chemistry, electric conductance of biological systems, diffusion processes, damping laws and rheology. Among the recent applications we mention areas like the modeling of dynamical systems in psychology and life sciences [4]. For details, see [4–10] and references therein.

The arbitrary (fractional) order integral operator is a singular integral operator, and the arbitrary (fractional) order differential operator, is a singular integro-differential operator. They generalize the integral and differential operators of integer orders. In the theory of one-dimensional fractional integration it is well known that the *Weyl* periodic fractional integral of a 2π-periodic function $f(x)$ coincides with the properly interpreted *Riemann–Liouville* fractional integral of f [11]. In fact the periodic Weyl–Riesz fractional kernel $\sum_{n=-\infty}^{\infty} \frac{\exp(inx)}{(in)^\alpha}$ is the periodization of the Riemann–Liouville kernel $\frac{x^{\alpha-1}}{\Gamma(\alpha)}$.

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The periodization, at least for fractional integration, appears in the paper H. Weyl [12]. A general idea of the periodicity of a function given on a real line, is presented in the book Zygmund [13]. To keep a function periodic, one should introduce the fractional integration as periodic convolution

$$\frac{1}{2\pi}\int_0^{2\pi}\psi^\alpha(t-\tau)f(\tau)\,d\tau,$$

with a suitable periodic kernel $\psi^\alpha(x)$ playing the same role as the power function $\frac{x^{\alpha-1}}{\Gamma(\alpha)}$ does in the nonperiodic case. *Weyl* [12] introduced the fractional integration keeping periodicity in the convolution form

$$W_\pm^\alpha f(t)=\frac{1}{2\pi}\int_0^{2\pi}\psi_\pm^\alpha(t-\tau)f(\tau)\,d\tau,\quad \alpha>0,$$

where

$$\psi_\pm^\alpha(\tau):=\sum_{n=-\infty}^{\infty}\frac{e^{inx}}{(\pm in)^\alpha}=2\sum_{n=1}^{\infty}\frac{\cos(n\tau\mp\alpha\pi/2)}{n^\alpha},$$

the dash indicating that the term $n=0$ is omitted and the sign $\pm$ corresponding to the left- and right-hand side forms of fractional integration (see [9,11]). Also the *Weyl–Riesz* fractional integration operator of periodic functions takes the form

$$W^\alpha f(t)=\frac{1}{2\pi}\int_0^{2\pi}\psi^\alpha(\tau)f(t-\tau)\,d\tau,$$

where

$$\psi^\alpha(\tau)=2\sum_{n=1}^{\infty}\frac{\cos n\tau}{n^\alpha}\quad\text{and}\quad 0\leqslant\tau\leqslant 2\pi.$$

For more details on the mathematical properties of the *Weyl–Riesz* fractional integration operator, see [9,11].

The objective of this paper is to apply the fixed point theorems given by Dhage [14,15] to prove the existence of periodic solutions for the nonlinear integral equation

$$u(t)=\big[g\big(t,u(t)\big)\big]\left[\frac{1}{2\pi}\int_0^{2\pi}\psi^\alpha(\tau)f\big(t-\tau,u(t-\tau)\big)\,d\tau\right],\quad 0<\alpha<1,\ t\in J:=[0,2\pi],\tag{1}$$

where $f:J\times\mathbb{R}\to\mathbb{R}$ and $g:J\times\mathbb{R}\to\mathbb{R}-\{0\}$ are periodic functions of period 2π. Evidently, for $\alpha=1$, Eq. (1) turns into a nonlinear integral equation of the Hammerstein type. The additional parameter α allows more generality and that can be varied to obtain various responses. Finally, we should mention that the *Riemann–Liouville* fractional integro-differentiation does not preserve periodicity of functions, as is well known.

In order to proceed, we give some definitions and theorems which are used further in this paper.

Definition 1.1. (See [3].) A mapping $p:J\times\mathbb{R}\to\mathbb{R}$ is said to be *Carathéodory* if

(i) $t\to p(t,u)$ is measurable for each $u\in\mathbb{R}$,
(ii) $u\to p(t,u)$ is continuous a.e. for $t\in J$.

A *Carathéodory* function $p(t,u)$ is called $L^1(J,\mathbb{R})$-*Carathéodory* if

(iii) for each number $r>0$ there exists a function $h_r\in L^1(J,\mathbb{R})$ such that $|p(t,u)|\leqslant h_r(t)$ a.e. $t\in J$ for all $u\in\mathbb{R}$ with $|u|\leqslant r$.

A *Carathéodory* function $p(t,u)$ is called $L^1_X(J,\mathbb{R})$-*Carathéodory* if

(iv) there exists a function $h \in L^1(J,\mathbb{R})$ such that $|p(t,u)| \leqslant h(t)$ a.e. $t \in J$ for all $u \in \mathbb{R}$, where h is called the bounded function of p.

Definition 1.2. Let $\mathcal{X}$ be a Banach algebra with norm $\|.\|$. A mapping $A : \mathcal{X} \to \mathcal{X}$ is called $\mathcal{D}$-Lipschitz if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \leqslant \psi\big(\|x - y\|\big), \quad \forall x, y \in \mathcal{X} \text{ with } \psi(0) = 0.$$

Definition 1.3. A Banach space $\mathcal{X}$ endowed with a closed cone K is an ordered Banach space $(\mathcal{X}, K)$ with a partial order $\leqslant$ in $\mathcal{X}$ as follows: $x \leqslant y$ if $y - x \in K$. And it is called normal if the norm $\|.\|$ is monotone increasing.

Theorem 1.1. *(See [3].) Let $\mathcal{X}$ be a Banach algebra and let $A, B : \mathcal{X} \to \mathcal{X}$ be two operators satisfying*

(a) *A is a $\mathcal{D}$-Lipschitz with a $\mathcal{D}$-function ψ.*
(b) *B is compact and continuous.*
(c) *$M\psi(r) < r$ whenever $r > 0$, where $M = \|B(\mathcal{X})\| = \sup\{B(x)\colon x \in \mathcal{X}\}$.*

Then either

(i) *the equation $\lambda AxBx = x$ has a solution for $\lambda = 1$, or*
(ii) *the set $E = \{u \in \mathcal{X}\colon \lambda AuBu = u,\ 0 < \lambda < 1\}$ is unbounded.*

Corollary 1.1. *(See [3].) Let $\mathcal{X}$ be a Banach algebra and let $A, B : \mathcal{X} \to \mathcal{X}$ be two operators satisfying*

(a) *A is a Lipschitz with a Lipschitz constant β.*
(b) *B is compact and continuous.*
(c) *$\beta M < 1$, where $M = \|B(\mathcal{X})\| = \sup\{B(x)\colon x \in \mathcal{X}\}$.*

Then either

(i) *the equation $\lambda AxBx = x$ has a solution for $\lambda = 1$, or*
(ii) *the set $E = \{u \in \mathcal{X}\colon \lambda AuBu = u,\ 0 < \lambda < 1\}$ is unbounded.*

Theorem 1.2. *(See [15].) Let K be a cone in a Banach algebra $\mathcal{X}$ and let $\underline{u}, \overline{u} \in \mathcal{X}$. Suppose that $A, B : [\underline{u}, \overline{u}] \to K$ be two operators such that*

(a) *A is a Lipschitz with a Lipschitz constant β.*
(b) *B is completely continuous.*
(c) *$AxBx \in [\underline{u}, \overline{u}]$ for each $x \in [\underline{u}, \overline{u}]$.*
(d) *A and B are nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $AxBx = x$ has a least and a greatest positive solutions in $[\underline{u}, \overline{u}]$, whenever $\beta M < 1$, where $M = \|B([\underline{u}, \overline{u}])\| = \sup\{B(x)\colon x \in [\underline{u}, \overline{u}]\}$.

Theorem 1.3. *(See [15].) Let K be a cone in a Banach algebra $\mathcal{X}$ and let $\underline{u}, \overline{u} \in \mathcal{X}$. Suppose that $A, B : [\underline{u}, \overline{u}] \to K$ be two operators such that*

(a) *A is a Lipschitz with a Lipschitz constant β.*
(b) *B is totally bounded (maps a bounded subset of $\mathcal{X}$ into the relatively compact subset of $\mathcal{X}$) and nondecreasing.*
(c) *$AxBy \in [\underline{u}, \overline{u}]$ for each $x, y \in [\underline{u}, \overline{u}]$.*

Further if the cone K is positive and normal, then the operator equation $AxBx = x$ has a least and a greatest positive solutions in $[\underline{u}, \overline{u}]$ whenever $\beta M < 1$, where $M = \|B([\underline{u}, \overline{u}])\| = \sup\{B(x)\colon x \in [\underline{u}, \overline{u}]\}$.

Remark 1.1. Note that in Theorems 1.2, 1.3, the operators A, B are monotone increasing and there exist elements $\underline{u}, \overline{u} \in \mathcal{B}$ such that $\underline{u} \leqslant A\underline{u}B\underline{u}$ and $A\overline{u}B\overline{u} \leqslant \overline{u}$.

2. An existence result

In this section, we discuss the existence of positive 2π-periodic solutions for Eq. (1) in view of Theorem 1.1. Assume that $\mathcal{B} := C[J, \mathbb{R}]$ is the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\| = \sup\{|u(t)|\colon t \in J\},$$

for each $u \in \mathcal{B}$. Again define a multiplication (.) by $(u.v)(t) = u(t)v(t)$ for all $t \in J$. Then $\mathcal{B} := C[J, \mathbb{R}]$ is a Banach algebra with respect to the above norm and multiplication in it. Let $\mathcal{L} := L^1[J, \mathbb{R}]$ denotes the Banach space of measurable functions $u : J \to \mathbb{R}$ which are Lebesgue integrable normed by

$$\|u\|_{L^1} = \int_0^T |u(t)|\, dt,$$

for each $u \in \mathcal{L}$.

Also, we need the following hypotheses in the sequel.

- (H1) $f : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is a periodic function of period 2π. Moreover, assume that f is $L^1_X(J, \mathbb{R})$-*Carathéodory* with bounded 2π-periodic $h(t) \in L^1(J, \mathbb{R}^+)$ such that $2\rho_\alpha \|h\|_{L^1} := M < \infty$, where $\rho_\alpha := \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$.
- (H2) $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is a periodic function of period 2π. Moreover, assume that g is L-*Lipschitz* in its second argument with $0 < ML < 1$.
- (H3) Denotes that $\phi := \sup_{t \in J} |g(t, 0)|$.

Now, we are in a position to state and prove our main result in this section.

Theorem 2.1. *Let the assumptions* (H1)–(H3) *hold. Then Eq.* (1) *has at least one periodic solution on J.*

Proof. Define two operators $A, B : \mathcal{B} \to \mathcal{B}$ where $\mathcal{B} = C[J, \mathbb{R}]$ as follows

$$Au(t) := g\big(t, u(t)\big), \quad \text{for all } t \in J, \tag{2}$$

$$Bu(t) := \frac{1}{2\pi} \int_0^{2\pi} \psi^\alpha(\tau) f\big(t - \tau, u(t - \tau)\big)\, d\tau, \quad \text{for all } t \in J. \tag{3}$$

Then we can rewrite Eq. (1) as an operator equation

$$u(t) = Au(t)Bu(t), \quad t \in J. \tag{4}$$

Our aim is to show that the operators A, B satisfy all the hypotheses of Corollary 1.1. First we proceed to prove that the operator A is Lipschitz on $\mathcal{B}$. Let $u, v \in \mathcal{B}$, then we have

$$\begin{aligned} |Au(t) - Av(t)| &\leqslant \big|g\big(t, u(t)\big) - g\big(t, v(t)\big)\big| \\ &\leqslant L\|u - v\|_{\mathcal{B}}, \end{aligned}$$

taking the supremum over t we obtain

$$\|Au - Av\| \leqslant L\|u - v\|_{\mathcal{B}}, \quad \text{for all } u, v \in \mathcal{B}.$$

Thus A is Lipschitz on $\mathcal{B}$ with Lipschitz constant L. Next we show that B is completely continuous. By (H2) we have that f is a continuous function on $\mathcal{B}$. Then consequently we obtain that $Bu(t)$ is continuous on $\mathcal{B}$. We only need to prove that $B(\mathcal{B})$ is uniformly bounded and equicontinuous in $\mathcal{B}$. Let S be a bounded set in $\mathcal{B}$,

$$\begin{aligned}
\left|Bu(t)\right| &\leqslant \frac{2}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{|\cos n\tau|}{n^\alpha}\left|f\big(t-\tau,u(t-\tau)\big)\right|d\tau \\
&\leqslant \frac{2}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{1}{n^\alpha}\left|f\big(t-\tau,u(t-\tau)\big)\right|d\tau \\
&\leqslant \frac{2}{2\pi}\int_0^{2\pi}\rho_\alpha h(t-\tau)\,d\tau \\
&\leqslant \frac{2\rho_\alpha\|h\|_{L^1}}{2\pi}\int_0^{2\pi}d\tau = M,
\end{aligned}$$

taking the supremum over t we obtain $\|Bu\| \leqslant M$ for all u in S. Thus $B(\mathcal{B})$ is uniformly bounded set in $\mathcal{B}$. Now we show that $B(\mathcal{B})$ is equicontinuous set. Let $t_1, t_2 \in J$, then for any $u \in \mathcal{B}$ we have

$$\begin{aligned}
\left|Bu(t_1)-Bu(t_2)\right| &\leqslant \frac{2}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{|\cos n\tau|}{n^\alpha}\left|f\big(t_1-\tau,u(t_1-\tau)\big)-f\big(t_2-\tau,u(t_2-\tau)\big)\right|d\tau \\
&\leqslant \frac{2}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{1}{n^\alpha}\left|f\big(t_1-\tau,u(t_1-\tau)\big)+f\big(t_2-\tau,u(t_2-\tau)\big)\right|d\tau \\
&\leqslant \frac{2}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{1}{n^\alpha}\left[\left|f\big(t_1-\tau,u(t_1-\tau)\big)\right|+\left|f\big(t_2-\tau,u(t_2-\tau)\big)\right|\right]d\tau \\
&\leqslant \frac{2}{2\pi}\int_0^{2\pi}\rho_\alpha\big[h(t_1-\tau)+h(t_2-\tau)\big]\,d\tau \leqslant \frac{4\rho_\alpha\|h\|_{L^1}}{2\pi}\int_0^{2\pi}d\tau = 2M,
\end{aligned}$$

which is independent of u. Hence $B(\mathcal{B})$ is equicontinuous set, then $B(\mathcal{B})$ is relatively compact by Arzela–Ascoli theorem. Thus all the conditions of Corollary 1.1, be satisfied, then either (i) or (ii) holds. We show that (ii) cannot hold. Let $u \in \mathcal{B}$ be any solution for (1), then for all $\lambda \in (0,1)$,

$$u(t) = \lambda\big[g\big(t,u(t)\big)\big]\left[\frac{1}{2\pi}\int_0^{2\pi}\psi^\alpha(\tau)f\big(t-\tau,u(t-\tau)\big)\,d\tau\right], \quad t \in J.$$

Therefore for $0 < \lambda < 1$ we have

$$\begin{aligned}
\left|u(t)\right| &\leqslant \lambda\left|\big[g\big(t,u(t)\big)\big]\right|\left|\left[\frac{1}{2\pi}\int_0^{2\pi}\psi^\alpha(\tau)f\big(t-\tau,u(t-\tau)\big)\,d\tau\right]\right| \\
&\leqslant \lambda\left[\left|g\big(t,u(t)\big)-g(t,0)\right|+\left|g(t,0)\right|\right]\left[\frac{1}{2\pi}\int_0^{2\pi}\psi^\alpha(\tau)\left|f\big(t-\tau,u(t-\tau)\big)\right|d\tau\right] \\
&\leqslant \big[L\left|u(t)\right|+\phi\big]M.
\end{aligned}$$

Hence we obtain that $\|u\|_{\mathcal{B}} \leqslant \frac{\phi M}{1-ML}$ where $ML < 1$. Thus (ii) of Corollary 1.1 does not hold. Therefore the operator Eq. (4) and consequently Eq. (1) has a 2π-solution on J. This completes the proof. □

3. Extremal results

In this section we shall apply Theorems 1.2 and 1.3 to prove the existence of maximal and minimal solutions for Eq. (1). For this purpose, we need the following definitions

Definition 3.1. Let m be a solution of Eq. (1) in J, then m is said to be a maximal solution of (1) if for every solution u of (1) existing on J the inequality $u(t) \leqslant m(t), t \in J$ holds. A minimal solution may be define similarly by reversing the last inequality.

Definition 3.2. A function $\underline{u} \in \mathcal{B}$ is called a lower solution of Eq. (1) on J if

$$\underline{u}(t) \leqslant \left[g\big(t, u(t)\big)\right]\left[\frac{1}{2\pi}\int_0^{2\pi} \psi^\alpha(\tau) f\big(t-\tau, u(t-\tau)\big)\,d\tau\right], \quad \text{a.e. } t \in J.$$

Again a function $\bar{u} \in \mathcal{B}$ is called an upper solution of Eq. (1) on J if

$$\bar{u}(t) \geqslant \left[g\big(t, u(t)\big)\right]\left[\frac{1}{2\pi}\int_0^{2\pi} \psi^\alpha(\tau) f\big(t-\tau, u(t-\tau)\big)\,d\tau\right], \quad \text{a.e. } t \in J.$$

Also, we need the following assumptions in the sequel.

(H4) The 2π-periodic, nondecreasing function $f : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^+$ is $L^1_X(J, \mathbb{R})$-*Carathéodory* with bounded 2π-periodic $h(t) \in L^1(J, \mathbb{R}^+)$ such that $2\rho_\alpha \|h\|_{L^1} := M < \infty$, where $\rho_\alpha := \sum_{n=1}^\infty \frac{1}{n^\alpha}$.
(H5) The 2π-periodic, nondecreasing function $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^+$ is *L-Lipschitz* in its second argument.
(H6) The Eq. (1) has a lower and upper solutions.

Theorem 3.1. *Let the assumptions* (H4)–(H6) *hold. If* $0 < ML < 1$, *then Eq.* (1) *has a minimal and maximal positive* 2π*-periodic solutions on* J.

Proof. Define an order relation $\leqslant$ by the cone K defined by $K = \{u \in \mathcal{B}\colon u(t) \geqslant 0,\ 0 \leqslant t \leqslant 2\pi\}$. Clearly K is positive normal cone in $\mathcal{B}$ (see [16]). Consider a closed interval $[\underline{u}, \bar{u}]$ in $\mathcal{B}$ which is well defined in view of hypothesis (H6). Define two operators A, B as in Eqs. (2) and (3), respectively. Then we can put Eq. (1) as an operator equation in Banach algebra $\mathcal{B}$: $u(t) = Au(t)Bu(t)$, $t \in J$. Thus we have $A, B : [\underline{u}, \bar{u}] \to K$. Since the cone K is normal, then $[\underline{u}, \bar{u}]$, with $\underline{u} < \bar{u}$, is a normal bounded set in $\mathcal{B}$. As in the proof of Theorem 2.1, we can show that A is a Lipschitz and B is completely continuous on $[\underline{u}, \bar{u}]$. By assumptions (H4) and (H5) we have that A and B are nondecreasing operators on $[\underline{u}, \bar{u}]$. Because for $u, v \in [\underline{u}, \bar{u}]$ such that $u \leqslant v$. Since f and g are nondecreasing, then we obtain

$$Au(t) = g(t, u) \leqslant g(t, v) = Av(t), \quad \forall t \in J.$$

Similarly

$$Bu(t) = \frac{1}{2\pi}\int_0^{2\pi} \psi^\alpha(\tau) f\big(t-\tau, u(t-\tau)\big)\,d\tau \leqslant \frac{1}{2\pi}\int_0^{2\pi} \psi^\alpha(\tau) f\big(t-\tau, v(t-\tau)\big)\,d\tau = Bv(t).$$

Again since Eq. (1) has a lower and upper solutions, then for all $u \in [\underline{u}, \bar{u}]$ we obtain

$$\underline{u}(t) \leqslant \left[g\big(t, \underline{u}(t)\big)\right]\left[\frac{1}{2\pi}\int_0^{2\pi} \psi^\alpha(\tau) f\big(t-\tau, \underline{u}(t-\tau)\big)\,d\tau\right]$$

$$\leqslant \big[g\big(t,u(t)\big)\big]\left[\frac{1}{2\pi}\int_0^{2\pi}\psi^\alpha(\tau)f\big(t-\tau,u(t-\tau)\big)\,d\tau\right]$$
$$\leqslant \big[g\big(t,\bar{u}(t)\big)\big]\left[\frac{1}{2\pi}\int_0^{2\pi}\psi^\alpha(\tau)f\big(t-\tau,\bar{u}(t-\tau)\big)\,d\tau\right]$$
$$\leqslant \bar{u}(t),$$

as a result $\underline{u}(t) \leqslant Au(t)Bu(t) \leqslant \bar{u}(t)$ for all $t \in J$ and $u \in [\underline{u}, \bar{u}]$. Hence $AuBu \in [\underline{u}, \bar{u}]$ for all $u \in [\underline{u}, \bar{u}]$. Moreover, we have

$$\begin{aligned}\big\|B\big([\underline{u},\bar{u}]\big)\big\| &= \sup\big\{\|Bu\|\colon u \in [\underline{u},\bar{u}]\big\}\\ &\leqslant \sup\left\{\frac{2}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{|\cos n\tau|}{n^\alpha}\big|f\big(t-\tau,u(t-\tau)\big)\big|\,d\tau,\ u\in[\underline{u},\bar{u}]\right\}\\ &\leqslant \sup\left\{\frac{2}{2\pi}\int_0^{2\pi}\rho_\alpha h(t-\tau)\,d\tau,\ u\in[\underline{u},\bar{u}]\right\}\\ &\leqslant \frac{2\rho_\alpha\|h\|_{L^1}}{2\pi}\int_0^{2\pi}d\tau = M.\end{aligned}$$

Since $ML < 1$, we apply Theorem 1.2 to the operator equation $AuBu = u$ to yield that (1) has a minimal and maximal positive 2π-periodic solutions on J. The proof therefore is complete. □

For discontinuous case we need the following definition

Definition 3.3. (See [3].) A mapping $p : J \times \mathbb{R} \to \mathbb{R}$ is said to be *Chandrabhan* if

(i) $t \mapsto p(t,u)$ is measurable for each $u \in \mathbb{R}$,
(ii) $u \mapsto p(t,u)$ is nondecreasing almost everywhere for $t \in J$.

Again a *Chandrabhan* function $p(t,u)$ is called L^1-*Chandrabhan* if

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J,R)$ such that

$$\big|p(t,u)\big| \leqslant h_r(t), \quad \text{a.e. } t \in J,\ \forall u \in \mathbb{R} \text{ with } |u| \leqslant r.$$

Finally an L^1-*Chandrabhan* function $p(t,u)$ is called L^1_X-*Chandrabhan* if
(iv) there exists a function $h \in L^1(J,\mathbb{R})$ (bound function of p) such that

$$\big|p(t,u)\big| \leqslant h(t), \quad \text{a.e. } t \in J,\ \forall u \in \mathbb{R}.$$

In order to apply Theorem 1.3 on Eq. (1), we need the following assumption:

(H7) The 2π-periodic function f is L^1_X-*Chandrabhan.*

Theorem 3.2. *Let the assumptions* (H5)–(H7) *hold. If* $0 < ML < 1$, *then Eq.* (1) *has a minimal and maximal positive* 2π*-periodic solutions on* J.

Proof. Define an order relation $\leqslant$ by the cone K defined by $K = \{u \in \mathcal{B}\colon u(t) \geqslant 0,\ 0 \leqslant t \leqslant 2\pi\}$. Clearly K is positive normal cone in $\mathcal{B}$ (see [16]). Consider a closed interval $[\underline{u}, \bar{u}]$ in $\mathcal{B}$ which is well defined in view of hypothesis (H6).

Define two operators A, B as in Eqs. (2) and (3), respectively. Then we can put Eq. (1) as an operator equation in Banach algebra $\mathcal{B}$: $u(t) = Au(t)Bu(t)$, $t \in J$. Hence we have $A, B : [\underline{u}, \overline{u}] \to K$. Since the cone K is normal, then $[\underline{u}, \overline{u}]$, with $\underline{u} < \overline{u}$, is a normal bounded set in $\mathcal{B}$. As in the proof of Theorem 2.1, we can show that A is a Lipschitz operator. Next we show that B is totally bounded operator on $[\underline{u}, \overline{u}]$. We shall show that B is uniformly bounded and equicontinuous set $\mathcal{B}$ for any subset S of $[\underline{u}, \overline{u}]$,

$$\begin{aligned}
\big|Bu(t)\big| &\leqslant \frac{2}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{|\cos n\tau|}{n^\alpha}\big|f\big(t-\tau, u(t-\tau)\big)\big|\,d\tau \\
&\leqslant \frac{2}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{1}{n^\alpha}\big|f\big(t-\tau, u(t-\tau)\big)\big|\,d\tau \\
&\leqslant \frac{2}{2\pi}\int_0^{2\pi}\rho_\alpha h(t-\tau)\,d\tau \\
&\leqslant \frac{2\rho_\alpha\|h\|_{L^1}}{2\pi}\int_0^{2\pi}d\tau = M,
\end{aligned}$$

taking the supremum over t we obtain $\|Bu\| \leqslant M$ for all u in S. Thus $B(\mathcal{B})$ is uniformly bounded set in $\mathcal{B}$. Now we show that $B(\mathcal{B})$ is equicontinuous set. Let $t_1, t_2 \in J$, then for any $u \in \mathcal{B}$ we have

$$\begin{aligned}
\big|Bu(t_1) - Bu(t_2)\big| &\leqslant \frac{2}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{|\cos n\tau|}{n^\alpha}\big|f\big(t_1-\tau, u(t_1-\tau)\big) - f\big(t_2-\tau, u(t_2-\tau)\big)\big|\,d\tau \\
&\leqslant \frac{2}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{1}{n^\alpha}\big|f\big(t_1-\tau, u(t_1-\tau)\big) + f\big(t_2-\tau, u(t_2-\tau)\big)\big|\,d\tau \\
&\leqslant \frac{2}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{1}{n^\alpha}\big[\big|f\big(t_1-\tau, u(t_1-\tau)\big)\big| + \big|f\big(t_2-\tau, u(t_2-\tau)\big)\big|\big]\,d\tau \\
&\leqslant \frac{2}{2\pi}\int_0^{2\pi}\rho_\alpha\big[h(t_1-\tau) + h(t_2-\tau)\big]\,d\tau \leqslant \frac{4\rho_\alpha\|h\|_{L^1}}{2\pi}\int_0^{2\pi}d\tau = 2M,
\end{aligned}$$

which is independent of u. Hence $B(\mathcal{B})$ is equicontinuous set, then $B(\mathcal{B})$ is relatively compact by Arzela–Ascoli theorem. Thus B is totally bounded. It is clear that B is nondecreasing operator this comes from (H7) and Definition 3.3. Again since Eq. (1) has a lower and upper solutions, then for all $u, v \in [\underline{u}, \overline{u}]$ we obtain

$$\begin{aligned}
\underline{u}(t) &\leqslant \big[g\big(t, \underline{u}(t)\big)\big]\left[\frac{1}{2\pi}\int_0^{2\pi}\psi^\alpha(\tau)f\big(t-\tau, \underline{u}(t-\tau)\big)\,d\tau\right] \\
&\leqslant \big[g\big(t, u(t)\big)\big]\left[\frac{1}{2\pi}\int_0^{2\pi}\psi^\alpha(\tau)f\big(t-\tau, v(t-\tau)\big)\,d\tau\right] \\
&\leqslant \big[g\big(t, \overline{u}(t)\big)\big]\left[\frac{1}{2\pi}\int_0^{2\pi}\psi^\alpha(\tau)f\big(t-\tau, \overline{u}(t-\tau)\big)\,d\tau\right] \\
&\leqslant \overline{u}(t),
\end{aligned}$$

as a result $\underline{u}(t) \leqslant Au(t)Bv(t) \leqslant \bar{u}(t)$ for all $t \in J$ and $u, v \in [\underline{u}, \bar{u}]$. Hence $AuBv \in [\underline{u}, \bar{u}]$ for all $u \in [\underline{u}, \bar{u}]$. Since $ML < 1$, we apply Theorem 1.3 to the operator equation $AuBu = u$ to yield that (1) has a minimal and maximal positive 2π-periodic solutions on J. The proof is complete. □

4. An application

The most important application of Eq. (1) is the integral equation of the Hammerstein type which takes the form

$$u(t) = g\big(t, u(t)\big) \int_0^\infty K(t,\tau) f\big(\tau, u(\tau)\big)\, d\tau, \quad t \geqslant 0.$$

In the case of bounded interval

$$u(t) = g\big(t, u(t)\big) \int_a^b K(t,\tau) f\big(\tau, u(\tau)\big)\, d\tau, \quad t \in [a,b].$$

The above integral equations have many applications to real world problems. For example in vehicular traffic theory, biology and queuing theory, radiative transfer theory, neutron transport theory and kinetic theory of gases [17]. Thus the Hammerstein type equations create a generalization of several kinds of integral equations. The fractional integral equation of Hammerstein type for periodic functions takes the form

$$u(t) = \big[g\big(t, u(t)\big)\big]\left[\frac{1}{2\pi}\int_0^{2\pi} \psi^\alpha(\tau) f\big(t-\tau, u(t-\tau)\big)\, d\tau\right], \quad 0 < \alpha < 1,\ t \in J = [0, 2\pi].$$

Also the integral equation of the Hammerstein type takes the form

$$u(t) = h(t) + g\big(t, u(t)\big) \int_0^\infty K(t,\tau) f\big(\tau, u(\tau)\big)\, d\tau, \quad t \geqslant 0,$$

and in the case of bounded interval

$$u(t) = h(t) + g\big(t, u(t)\big) \int_a^b K(t,\tau) f\big(\tau, u(\tau)\big)\, d\tau, \quad t \in [a,b].$$

In this section we study the existence of periodic solutions for the fractional integral equation of Hammerstein type of the form

$$u(t) = h(t) + \big[g\big(t, u(t)\big)\big]\left[\frac{1}{2\pi}\int_0^{2\pi} \psi^\alpha(\tau) f\big(t-\tau, u(t-\tau)\big)\, d\tau\right], \quad 0 < \alpha < 1,\ t \in J = [0, 2\pi], \tag{5}$$

where $h(t) \in L^1(J, \mathbb{R})$ is 2π-periodic function.

Theorem 4.1. *Let the assumptions* (H1)–(H3) *hold. Then Eq.* (5) *has a periodic solution.*

Proof. In order to show that (5) has a solution we only need to prove that the operator

$$(Pu) := h(t) + \big[g\big(t, u(t)\big)\big]\left[\frac{1}{2\pi}\int_0^{2\pi} \psi^\alpha(\tau) f\big(t-\tau, u(t-\tau)\big)\, d\tau\right], \quad 0 < \alpha < 1,\ t \in J = [0, 2\pi],$$

has a fixed point

$$\begin{aligned}\big|(Pu)(t)\big| &\leqslant \big|h(t)\big| + \big[\big|g\big(t,u(t)\big)-g(t,0)\big| + \big|g(t,0)\big|\big]\left[\frac{1}{2\pi}\int_0^{2\pi}\psi^{\alpha}(\tau)\big|f\big(t-\tau,u(t-\tau)\big)\big|\,d\tau\right] \\ &\leqslant \|h\|_{L^1} + \big[L\big|u(t)\big| + \phi\big]M,\end{aligned}$$

that is

$$\big\|P(u)\big\|_{\mathcal{B}} = \sup_{t\in J}\big|P\big(u(t)\big)\big| \leqslant \frac{\|h\|_{L^1} + \phi M}{1-LM} := r,$$

that is $P : B_r \to B_r$. Then P maps B_r into itself. In fact, P maps the convex closure of $P[B_r]$ into itself. Since f, g are bounded on B_r, $P[B_r]$ is equicontinuous and the Schauder fixed point theorem shows that P has a fixed point $u(t) \in \mathbb{B}$, such that $(Pu)(t) = u(t)$, which is corresponding to the solution of (5). □

Remark 4.1. The same argument of Theorem 4.1 holds for the fractional integral equation of Hammerstein type

$$u(t) = h(t) + \big[g\big(t,u(t)\big)\big]\left[\frac{1}{2\pi}\int_0^{2\pi}\psi^{\alpha}(t-\tau)f\big(\tau,u(\tau)\big)\,d\tau\right], \quad 0<\alpha<1,\ t\in J=[0,2\pi].$$

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