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Numerical methods for nonlinear partial differential equations of fractional order

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Abstract

In this article, we implement relatively new analytical techniques, the variational iteration method and the Adomian decomposition method, for solving nonlinear partial differential equations of fractional order. The fractional derivatives are described in the Caputo sense. The two methods in applied mathematics can be used as alternative methods for obtaining analytic and approximate solutions for different types of fractional differential equations. In these schemes, the solution takes the form of a convergent series with easily computable components. Numerical results show that the two approaches are easy to implement and accurate when applied to partial differential equations of fractional order.

Keywords: Variational iteration method; Adomian decomposition method; Lagrange multiplier; Fractional differential equation; Caputo fractional derivative

1. Introduction

Ordinary and partial differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. Consequently, considerable attention has been given to the solutions of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest [1–15]. Most nonlinear fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques (see [11–19]) must be used. The variational iteration method [12,14,19–28] and the Adomian decomposition method [29–32] are relatively new approaches to provide an analytical approximation to linear and nonlinear problems, and they are particularly valuable as tools for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization.

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The decomposition method has been used to obtain approximate solutions of a large class of linear or nonlinear differential equations [29,30]. Recently, the application of the method is extended for fractional differential equations [11–15,18]. The variational iteration method, which proposed by He [20–28], was successfully applied to autonomous ordinary and partial differential equations and other fields. Ji-Huan He [23] was the first to apply the variational iteration method to fractional differential equations. Recently Odibat and Momani [12] implemented the variational iteration method to solve nonlinear ordinary differential equations of fractional order.

The objective of the present paper is to extend the application of the variational iteration method to provide approximate solutions for initial value problems of nonlinear partial differential equations of fractional order and to make comparison with that obtained by Adomian decomposition method.

2. Definitions

For the concept of fractional derivative we will adopt Caputo's definition which is a modification of the Riemann–Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(>\mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geqslant 0$, of a function $f \in C_\mu$, $\mu \geqslant -1$, is defined as

$$\begin{aligned} J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t,\ \alpha > 0,\ x > 0 \\ J^0 f(x) &= f(x). \end{aligned} \tag{2.1}$$

Properties of the operator J^α can be found in [1,7,8], we mention only the following:

For $f \in C_\mu$, $\mu \geqslant -1$, $\alpha, \beta \geqslant 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by Caputo in his work on the theory of viscoelasticity [2].

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\mathrm{d}t, \tag{2.2}$$

for $m-1 < \alpha \leqslant m,\ m \in \mathbb{N},\ x > 0, f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 2.1. *If* $m-1 < \alpha \leqslant m$, $m \in N$ *and* $f \in C_\mu^m$, $\mu \geqslant -1$, *then*

$$D_*^\alpha J^\alpha f(x) = f(x) \tag{2.3}$$

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \tag{2.4}$$

3. Variational iteration method

The principles of the variational iteration method and its applicability for various kinds of differential equations are given in [12,20–28,33–35]. We consider the following time-fractional partial differential equation

$$D_{*t}^{\alpha}u(x,t) = f(u, u_x, u_{xx}) + g(x,t), \ \ m-1 < \alpha \leqslant m, \tag{3.1}$$

where $D_{*t}^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the Caputo fractional derivative of order α, $m \in \mathbf{N}$, f is a nonlinear function and g is the source function. The initial and boundary conditions associated with (3.1) are of the from

$$\begin{aligned} &u(x,0) = h(x), \ \ 0 < \alpha \leqslant 1, \\ &u(x,t) \to 0 \quad \text{as} \quad |x| \to \infty, \ \ t > 0, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &u(x,0) = h(x), \ \frac{\partial u(x,0)}{\partial t} = k(x), \ \ 1 < \alpha \leqslant 2, \\ &u(x,t) \to 0 \quad \text{as} \quad |x| \to \infty, \ \ t > 0. \end{aligned} \tag{3.3}$$

The correction functional for Eq. (3.1) can be approximately expressed as follows:

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^m}{\partial \xi^m}u_k(x,\xi) - f\left(\tilde{u}_k, (\tilde{u}_k)_x, (\tilde{u}_k)_{xx}\right) - g(x,\xi)\right)\mathrm{d}\xi, \tag{3.4}$$

where λ is a general Lagrange multiplier [36], which can be identified optimally via variational theory [21,26–28,36], here $(\tilde{u}_k)$, $(\tilde{u}_k)_x$, $(\tilde{u}_k)_{xx}$ are considered as restricted variations, i.e., $\delta\tilde{u}_n = 0$. Making the above functional stationary,

$$\delta u_{k+1}(x,t) = \delta u_k(x,t) + \delta\int_0^t \lambda(\xi)\left(\frac{\partial^m}{\partial \xi^m}u_k(x,\xi) - g(x,\xi)\right)\mathrm{d}\xi, \tag{3.5}$$

yields the following Lagrange multipliers

$$\begin{aligned} &\lambda = -1, \quad \text{for} \quad m = 1 \\ &\lambda = \xi - t, \quad \text{for} \quad m = 2. \end{aligned}$$

Therefore, for $m = 1$, we obtain the following iteration formula:

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left(\frac{\partial^{\alpha}}{\partial \xi^{\alpha}}u_k(x,\xi) - f\left(u_k, (u_k)_x, (u_k)_{xx}\right) - g(x,\xi)\right)\mathrm{d}\xi. \tag{3.6}$$

In this case, we begin with the initial approximation

$$u_0(x,t) = h(x). \tag{3.7}$$

For $m = 2$, we obtain the following iteration formula:

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^{\alpha}}{\partial \xi^{\alpha}}u_k(x,\xi) - f\left(u_k, (u_k)_x, (u_k)_{xx}\right) - g(x,\xi)\right)\mathrm{d}\xi. \tag{3.8}$$

In this case, we begin with the initial approximation

$$u_0(x,t) = h(x) + tk(x). \tag{3.9}$$

The correction functional (3.4) will give several approximations, and therefore the exact solution is obtained as

$$u(x,t) = \lim_{k\to\infty} u_k(x,t). \tag{3.10}$$

4. Decomposition method

The decomposition method requires that the nonlinear fractional differential Eq. (3.1) be expressed in terms of operator from as

$$D_{*t}^{\alpha}u(x,t) + Lu(x,t) + Nu(x,t) = g(x,t),\ x > 0, \tag{4.1}$$

where L is a linear operator which might include other fractional derivatives of order less than α, N is a non-linear operator which also might include other fractional derivatives of order less than α, $g(x,t)$ and D_{*t}^{α} are defined as in Eq. (3.1).

Applying the operator J^{α}, the inverse of the operator D_{*t}^{α}, to both sides of Eq. (4.1) yields

$$u(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J^{\alpha}g(x,t) - J^{\alpha}[Lu(x,t) + Nu(x,t)]. \tag{4.2}$$

The Adomian decomposition method [29–32] suggests the solution $u(x,t)$ be decomposed into the infinite series of components

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{4.3}$$

and the nonlinear function in Eq. (4.2) is decomposed as follows:

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{4.4}$$

where A_n are so-called the Adomian polynomials.

Substitution the decomposition series (4.3) and (4.4) into both sides of (4.2) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J^{\alpha}g(x,t) - J^{\alpha}\left[L\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \sum_{n=0}^{\infty} A_n\right]. \tag{4.5}$$

From this equation, the iterates are determined by the following recursive way

$$\begin{aligned} u_0(x,t) &= \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J^{\alpha}g(x,t), \\ u_1(x,t) &= -J^{\alpha}(Lu_0 + A_0), \\ u_2(x,t) &= -J^{\alpha}(Lu_1 + A_1), \\ &\vdots \\ u_{n+1}(x) &= -J^{\alpha}(Lu_n + A_n). \end{aligned} \tag{4.6}$$

The Adomian polynomial A_n can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [31]. The general form of formula for A_n Adomian polynomials is

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{n} \lambda^k u_k\right)\right]_{\lambda=0}. \tag{4.7}$$

This formula is easy to compute by using Mathematica software or by writing a computer code to get as many polynomials as we need in the calculation of the numerical as well as explicit solutions.

Finally, we approximate the solution $u(x,t)$ by the truncated series

$$\phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t) \quad \text{and} \quad \lim_{N\to\infty} \phi_N(x,t) = u(x,t). \tag{4.8}$$

However, in many cases the exact solution in a closed form may be obtained. Moreover, the decomposition series solutions generally converge very rapidly. The convergence of the decomposition series has been investigated in [37–39]. They obtained some results about the speed of convergence of this method. In recent work of Abbaoui and Cherruault [39] have proposed a new approach of convergence of the decomposition series. The authors have given a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM in [39].

5. Numerical experiments

In this section we shall illustrate the two techniques by several examples. These examples are somewhat artificial in the sense that the exact answer, for the special cases $\alpha = 1$ or 2, is known in advance and the initial and boundary conditions are directly taken from this answer. Nonetheless, such an approach is needed to evaluate the accuracy of the analytical techniques and to examine the effect of varying the order of the time-fractional derivative on the behavior of the solution. All the results are calculated by using the symbolic calculus software Mathematica.

Example 5.1. Consider the nonlinear time-fractional advection partial differential equation

$$D_{*t}^{\alpha}u(x,t) + u(x,t)u_x(x,t) = x + xt^2, \quad t > 0,\ x \in R,\ 0 < \alpha \leqslant 1, \tag{5.1}$$

subject to the initial condition

$$u(x,0) = 0. \tag{5.2}$$

According to the formula (3.6), the iteration formula for Eq. (5.1) is given by

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left(\frac{\partial^\alpha}{\partial \xi^\alpha}u_k(x,\xi) + u_k(x,\xi)(u_k)_x(x,\xi) - x - x\xi^2\right)\mathrm{d}\xi. \tag{5.3}$$

By the above variational iteration formula, begin with $u_0 = 0$, we can obtain the following approximations

$$u_0(x,t) = 0,$$

$$u_1(x,t) = x\left(t + \frac{t^3}{3}\right),$$

$$u_2(x,t) = x\left(2t + \frac{t^3}{3} - \frac{2t^5}{15} - \frac{t^7}{63} - \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{4-\alpha}}{\Gamma(5-\alpha)}\right),$$

$$\begin{aligned}u_3(x,t) = x\Bigg(&3t - \frac{2t^3}{3} - \frac{6t^5}{15} + \frac{2t^7}{45} + \frac{48t^9}{2835} - \frac{34t^{11}}{51,975} - \frac{4t^{13}}{12,285} - \frac{t^{15}}{59,535} - \frac{3t^{2-\alpha}}{\Gamma(3-\alpha)} \\ &+ \left(\frac{4}{(4-\alpha)\Gamma(3-\alpha)} - \frac{2}{\Gamma(5-\alpha)}\right)t^{4-\alpha} + \left(\frac{1}{\Gamma(6-2\alpha)} - \frac{1}{(5-2\alpha)\Gamma(3-\alpha)^2}\right)t^{5-2\alpha} \\ &+ 2\left(\frac{1}{3\Gamma(6-\alpha)\Gamma(3-\alpha)} + \frac{2}{(6-\alpha)\Gamma(5-\alpha)} + \frac{1}{3\Gamma(7-\alpha)}\right)t^{6-\alpha} - \frac{2t^{7-2\alpha}}{(7-2\alpha)\Gamma(3-\alpha)\Gamma(5-\alpha)} \\ &+ \left(\frac{1}{9\Gamma(9-\alpha)} + \frac{2}{3(8-\alpha)\Gamma(5-\alpha)} - \frac{4}{15(8-\alpha)\Gamma(3-\alpha)}\right)t^{8-\alpha} + \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} \\ &- \left(\frac{2}{63(10-\alpha)\Gamma(3-\alpha)} + \frac{4}{15(10-\alpha)\Gamma(5-\alpha)}\right)t^{10-\alpha} \\ &- \frac{2t^{12-\alpha}}{63(12-\alpha)\Gamma(5-\alpha)} - \frac{t^{9-2\alpha}}{(9-2\alpha)\Gamma(5-\alpha)^2}\Bigg),\end{aligned}$$

$$\vdots$$

and so on, in the same manner the rest of components of the iteration formula (5.3) can be obtained using the Mathematica package.

To solve the problem using the decomposition method, we simply substitute (5.1) and the initial conditions (5.2) into (4.6), to obtain the following recurrence relation

$$\begin{aligned} u_0(x,t) &= u(x,0) + J^{\alpha}(x + xt^2)) = x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right), \\ u_{j+1}(x,t) &= -J^{\alpha}(A_j),\ j \geqslant 0, \end{aligned} \tag{5.4}$$

where A_j are the Adomian polynomials for the nonlinear function $N = uu_x$. In view of (5.4), the first few components of the decomposition series are derived as follows:

$$\begin{aligned} u_0(x,t) &= x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right), \\ u_1(x,t) &= -x\left(\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} + \frac{4\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} + \frac{4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(\alpha+3)^2\Gamma(3\alpha+5)}\right), \\ u_2(x,t) &= 2x\left(\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} + \frac{8\Gamma(2\alpha+5)\Gamma(4\alpha+7)t^{5\alpha+6}}{\Gamma(\alpha+3)^3\Gamma(3\alpha+5)\Gamma(5\alpha+7)} + \cdots\right), \\ &\ \vdots \end{aligned}$$

and so on, in this manner the rest of components of the decomposition series can be obtained.

The first three terms of the decomposition series (4.3) are given by

$$u(x,t) = x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} - \frac{4\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} + \cdots\right).$$

Table 1 shows the approximate solutions for Eq. (5.1) obtained for different values of α using the decomposition method and the variational iteration method. The values of $\alpha = 1$ is the only case for which we know the exact solution $u(x,t) = xt$ and our approximate solution using the decomposition method is more accurate than the approximate solution obtained using the variational iteration method. It is to be noted that only the fourth-order term of the variational iteration solution and only three terms of the decomposition series were used in evaluating the approximate solutions for Table 1. It is evident, from (3.10) and (4.8), that the efficiency of these approaches can be dramatically enhanced by computing further terms or further components of $u(x,t)$ when the variational iteration method or the decomposition method are used.

Table 1
Numerical values when $\alpha = 0.5$, 0.75 and 1.0 for Eq. (5.1)

t	x	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$		
		u_{ADM}	u_{VIM}	u_{ADM}	u_{VIM}	u_{ADM}	u_{VIM}	u_{Exact}
0.2	0.25	0.112844	0.103750	0.078787	0.077933	0.050000	0.050309	0.050000
	0.50	0.225688	0.207499	0.157574	0.155865	0.100000	0.100619	0.100000
	0.75	0.311249	0.311249	0.236361	0.233798	0.150001	0.150928	0.150000
	1.0	0.451375	0.414999	0.315148	0.311730	0.200001	0.201237	0.200000
0.4	0.25	0.164004	0.172012	0.128941	0.134855	0.100023	0.101894	0.100000
	0.50	0.328008	0.344025	0.257881	0.269710	0.200046	0.203787	0.200000
	0.75	0.492011	0.516037	0.386821	0.404565	0.300069	0.305681	0.300000
	1.0	0.656015	0.688050	0.515762	0.539420	0.400092	0.407575	0.400000
0.6	0.25	0.243862	0.215641	0.177238	0.179990	0.150411	0.153094	0.150000
	0.50	0.487721	0.431283	0.354477	0.359979	0.300823	0.306188	0.300000
	0.75	0.731581	0.646924	0.531715	0.539969	0.451234	0.459282	0.450000
	1.0	0.975441	0.862566	0.7089541	0.719958	0.601646	0.612376	0.600000

Example 5.2. Consider the nonlinear time-fractional hyperbolic equation

$$D^{\alpha}_{*t}u(x,t) = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right), \quad t>0,\ x\in R,\ 1<\alpha\leqslant 2, \tag{5.5}$$

subject to the initial condition

$$u(x,0) = x^2, \quad u_t(x,0) = -2x^2. \tag{5.6}$$

According to the formula (3.8), the iteration formula for Eq. (5.5) is given by

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^{\alpha}}{\partial \xi^{\alpha}}u_k(x,\xi) - \frac{\partial}{\partial x}\left(u_k(x,\xi)\frac{\partial u_k(x,\xi)}{\partial x}\right)\right)\mathrm{d}\xi. \tag{5.7}$$

By the above variational iteration formula, begin with $u_0 = x^2 - 2tx^2$, we can obtain the following approximations

$$\begin{aligned}
u_0(x,t) &= x^2(1-2t),\\
u_1(x,t) &= x^2\left(1-2t+3t^2-4t^3+2t^4\right),\\
u_2(x,t) &= x^2\left(1-2t+6t^2-8t^3+7t^4-6t^5+\frac{174}{30}t^6-\frac{192}{42}t^7+\frac{168}{56}t^8-\frac{96}{72}t^9+\frac{24}{90}t^{10}\right)\\
&\quad + x^2\left(\frac{-6}{\Gamma(5-\alpha)}t^{4-\alpha}+\frac{24}{\Gamma(6-\alpha)}t^{5-\alpha}-\frac{48}{\Gamma(7-\alpha)}t^{6-\alpha}\right),\\
&\ \ \vdots
\end{aligned}$$

and so on, in the same manner the rest of components of the iteration formula (5.7) can be obtained using the Mathematica package.

To solve the problem using the decomposition method, we substitute (5.5) and the initial conditions (5.6) into (4.6), to obtain the recurrence relation

$$\begin{aligned}
u_0(x,t) &= u(x,0) + tu_x(x,0) = x^2(1-2t),\\
u_{j+1}(x,t) &= J^{\alpha}(A_j)_x, \quad j\geqslant 0,
\end{aligned} \tag{5.8}$$

where A_j are the Adomian polynomials for the nonlinear function $N = uu_x$. In view of (5.8), the first few components of the decomposition series are derived as follows:

$$\begin{aligned}
u_0(x,t) &= x^2(1-2t),\\
u_1(x,t) &= 6x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{4t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right),\\
u_2(x,t) &= 72x^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{4t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{8t^{2\alpha+2}}{\Gamma(2\alpha+3)}-\frac{2\Gamma(\alpha+2)t^{2\alpha+1}}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}\right)\\
&\quad + 72x^2\left(\frac{8\Gamma(\alpha+3)t^{2\alpha+2}}{\Gamma(\alpha+2)\Gamma(2\alpha+3)}-\frac{16\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)}\right),\\
&\ \ \vdots
\end{aligned}$$

and so on, in this manner the rest of components of the decomposition series can be obtained.

The first three terms of the decomposition series (4.3) are given by

$$u(x,t) = x^2(1-2t) + 6x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{4t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right) + 72x^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\right).$$

Table 2 shows the approximate solutions for Eq. (5.5) obtained for different values of α using the decomposition method and the variational iteration method. The values of $\alpha = 2$ is the only case for which we know the exact solution $u(x,t) = (x/t+1)^2$ and our approximate solution using the variational iteration method is

Table 2
Numerical values when $\alpha = 1.5$, 1.75 and 2.0 for Eq. (5.5)

t	x	$\alpha = 1.5$		$\alpha = 1.75$		$\alpha = 2.0$		
		u_{ADM}	u_{VIM}	u_{ADM}	u_{VIM}	u_{ADM}	u_{VIM}	u_{Exact}
0.2	0.25	0.0592832	0.047502	0.0497012	0.043403	0.0433951	0.043400	0.043403
	0.50	0.237133	0.190007	0.194805	0.184170	0.173580	0.173600	0.173611
	0.75	0.533549	0.427517	0.438311	0.414383	0.390556	0.390600	0.390625
	1.0	0.948532	0.760029	0.779220	0.736680	0.694321	0.694400	0.694444
0.4	0.25	0.0654119	0.041853	0.037742	0.037742	0.031567	0.031779	0.031888
	0.50	0.261647	0.167412	0.174992	0.150968	0.126268	0.127118	0.127551
	0.75	0.588707	0.376676	0.393732	0.339679	0.284103	0.286015	0.286990
	1.0	1.04659	0.669647	0.699969	0.603873	0.505072	0.508471	0.508471
0.6	0.25	0.063177	0.037722	0.381836	0.031457	0.022005	0.023665	0.024414
	0.50	0.252710	0.150888	0.152735	0.125829	0.088018	0.094660	0.097656
	0.75	0.568598	0.339499	0.343653	0.283114	0.198040	0.212984	0.219727
	1.0	1.01084	0.603553	0.610938	0.503314	0.352071	0.378638	0.390625

more accurate than the approximate solution obtained using the decomposition method. It is to be noted that only the third-order term of the variational iteration solution and only three terms of the decomposition series were used in evaluating the approximate solutions for Table 2. Of course the accuracy can be improved by computing more terms of the approximate solution.

Example 5.3. Consider the nonlinear time-fractional Fisher's equation

$$D_{*t}^{\alpha} u(x,t) = u_{xx}(x,t) + 6u(x,t)(1 - u(x,t)), \ t > 0, \ x \in R, \ 0 < \alpha \leqslant 1, \tag{5.9}$$

subject to the initial condition

$$u(x,0) = \frac{1}{(1+e^x)^2}. \tag{5.10}$$

According to the formula (3.6), the iteration formula for Eq. (5.9) is given by

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left(\frac{\partial^{\alpha}}{\partial \xi^{\alpha}} u_k(x,\xi) - (u_k)_{xx}(x,\xi) - 6u_k(x,\xi)(1 - u_k(x,\xi)) \right) d\xi. \tag{5.11}$$

By the above variational iteration formula, begin with $u_0 = 1/(1+e^x)^2$, we can obtain the following approximations

$$\begin{aligned}
u_0(x,t) &= \frac{1}{(1+e^x)^2}, \\
u_1(x,t) &= \frac{1}{(1+e^x)^2} + 10\frac{e^x}{(1+e^x)^3} t, \\
u_2(x,t) &= \frac{1}{(1+e^x)^2} + 10\frac{e^x}{(1+e^x)^3} t - 10\frac{e^x}{(1+e^x)^3}\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} \\
&\quad + \frac{1}{(1+e^x)^6}[-200e^{2x}t^3 + (50e^{4x} + 75e^{3x} - 25e^x)t^2 + (10e^{4x} + 30e^{3x} + 30e^{2x} + 10e^x)t],
\end{aligned}$$

$$
\begin{aligned}
u_3(x,t) = {} & \frac{1}{(1+e^x)^2}(1+6t) + \frac{10e^x}{(1+e^x)^3}\left[t + 3t^2 - \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)}\right] \\
& + \frac{t}{(1+e^x)^4}\left[4e^{2x} - 2e^x - 6\right] + \frac{10}{(1+e^x)^5}\left[4e^{3x} - 7e^{2x} - 11e^x\right]\left(\frac{t^2}{2} - \frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right) \\
& + \frac{1}{(1+e^x)^6}\Bigg[-300e^{2x}t^4 + \left(100e^{4x} + 150e^{3x} - 400e^{2x} - 50e^x\right)t^3 \\
& + \left(80e^{4x} + 165e^{3x} + 90e^{2x} + 5e^x\right)t^2 + \left(10e^{4x} + 30e^{3x} + 30e^{2x} + 10e^x\right)t \\
& - \left(10e^{4x} + 30e^{3x} + 30e^{2x}10e^x\right)\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - \left(100e^{4x} + 150e^{3x} - 50e^x\right)\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} \\
& + 1200e^{2x}\left(\frac{1}{\Gamma(5-\alpha)}\frac{1}{(4-\alpha)\Gamma(3-\alpha)}\right)t^{4-\alpha} - 600e^{2x}\frac{t^{5-2\alpha}}{(5-2\alpha)\Gamma(3-\alpha)^2}\Bigg] \\
& + \frac{1}{(1+e^x)^8}\Bigg[\left(-3200e^{4x} + 4400e^{3x} + 1600e^{2x}\right)\frac{t^4}{4} + \left(200e^{6x} - 425e^{5x} - 1600e^{4x} - 850e^{3x}\right. \\
& \left. + 400e^{2x} + 275e^x\right)\frac{t^3}{3} + \left(40e^{6x} + 50e^{5x} - 200e^{4x} - 500e^{3x} - 400e^{2x} - 110e^x\right)\frac{t^2}{2}\Bigg] \\
& - \frac{120e^x}{(1+e^x)^9}\Bigg[-200e^{2x}\left(\frac{t^5}{5} - \frac{t^{6-\alpha}}{(6-\alpha)\Gamma(3-\alpha)}\right) + \left(50e^{4x} + 75e^{3x} - 25e^x\right) \\
& \times\left(\frac{t^4}{4} - \frac{t^{5-\alpha}}{(5-\alpha)\Gamma(3-\alpha)}\right) + \left(10e^{4x} + 30e^{3x} + 30e^{2x} + 10e^x\right)\left(\frac{t^3}{3} - \frac{t^{4-\alpha}}{(4-\alpha)\Gamma(3-\alpha)}\right)\Bigg] \\
& - \frac{6}{(1+e^x)^{12}}\Bigg[40,000e^{4x}\frac{t^7}{7} + \left(-20,000e^{6x} - 30,000e^{5x} + 10,000e^{3x}\right)\frac{t^6}{6} + \left(2500e^{8x} + 7500e^{7x}\right. \\
& \left. + 1625e^{6x} - 14,500e^{5x} - 15,750e^{4x} - 4000e^{3x} + 625e^{2x}\right)\frac{t^5}{5} + \left(1000e^{8x} + 4500e^{7x} + 7500e^{6x}\right. \\
& \left. + 5000e^{5x} - 1500e^{3x} - 500e^{2x}\right)\frac{t^4}{4} + \left(100e^{8x} + 600e^{7x} + 1500e^{6x} + 2000e^{5x} + 1500e^{4x}\right. \\
& \left. + 600e^{3x} + 100e^{2x}\right)\frac{t^3}{3}\Bigg],
\end{aligned}
$$

$$\vdots$$

and so on, in the same manner the rest of components of the iteration formula (5.11) can be obtained using the Mathematica package.

To solve the problem using the decomposition method, we substitute (5.9) and the initial conditions (5.10) into (4.6), to obtain the following recurrence relation

$$
\begin{aligned}
u_0(x,t) &= u(x,0) = \frac{1}{(1+e^x)^2}, \\
u_{j+1}(x,t) &= J^\alpha\left((u_j)_{xx} + 6u_j - 6A_j\right), \quad j \geqslant 0,
\end{aligned}
\tag{5.12}
$$

where A_j are the Adomian polynomials for the nonlinear function $N = u^2$. In view of (5.12), the first few components of the decomposition series are derived as follows:

$$
\begin{aligned}
u_0(x,t) &= \frac{1}{(1+e^x)^2}, \\
u_1(x,t) &= \frac{10e^x}{(1+e^x)^3}\frac{t^\alpha}{\Gamma(\alpha+1)},
\end{aligned}
$$

Table 3
Numerical values when $\alpha = 0.5$, 0.75 and 1.0 for Eq. (5.9)

t	x	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$		
		u_{ADM}	u_{VIM}	u_{ADM}	u_{VIM}	u_{ADM}	u_{VIM}	u_{Exact}
0.1	0.25	0.946129	0.482361	0.488195	0.412450	0.317948	0.315940	0.316042
	0.50	0.843908	0.394446	0.405740	0.334514	0.250500	0.249926	0.250000
	0.75	0.715013	0.311106	0.324457	0.262103	0.190964	0.191606	0.191689
	1.0	0.576466	0.236710	0.249683	0.198407	0.140979	0.142411	0.142537
0.2	0.25	1.47532	0.746994	0.791250	0.617790	0.481199	0.459320	0.461284
	0.50	1.35983	0.653476	0.690142	0.536231	0.396941	0.386450	0.387456
	0.75	1.18098	0.548977	0.574404	0.448264	0.315266	0.315478	0.316042
	1.0	0.970076	0.441936	0.456647	0.359905	0.241175	0.249092	0.250000
0.3	0.25	1.96745	0.935741	1.12423	0.774999	0.681440	0.591179	0.604195
	0.50	1.845231	0.878473	1.00948	0.720112	0.581861	0.527635	0.534447
	0.75	1.622910	0.788974	0.859509	0.643697	0.475833	0.459719	0.461284
	1.0	1.345510	0.673844	0.695479	0.549294	0.372917	0.387025	0.387456

$$u_2(x,t) = \frac{50e^x(-1+2e^x)}{(1+e^x)^4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,t) = \frac{50e^x(5-6e^x-15e^{2x}+20e^{3x})\Gamma(\alpha+1)^2-12e^x\Gamma(2\alpha+1)}{(1+e^x)^6\Gamma(\alpha+1)^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$\vdots$$

and so on, in this manner the rest of components of the decomposition series can be obtained.

The first three terms of the decomposition series (5.4) are given by

$$u(x,t) = \frac{1}{(1+e^x)^2} + \frac{10e^x}{(1+e^x)^3}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{50e^x(-1+2e^x)}{(1+e^x)^4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Table 3 shows the approximate solutions for Eq. (5.9) obtained for different values of α using the decomposition method and the variational iteration method. The values of $\alpha = 1$ is the only case for which we know the exact solution $u(x,t) = 1/(1+e^{x-5t})^2$ and, as the previous example, our approximate solution using the variational iteration method is more accurate than the approximate solution obtained using the decomposition method. It is to be noted that only the fourth-order term of the variational iteration solution and only three terms of the decomposition series were used in evaluating the approximate solutions for Table 3.

6. Concluding remarks

The fundamental goal of this work has been to construct an approximate solution of nonlinear partial differential equations of fractional order. The goal has been achieved by using the variational iteration method and the Adomian decomposition method. The methods were used in a direct way without using linearization, perturbation or restrictive assumptions.

There are five important points to make here. First, the variational iteration method and the decomposition method provide the solutions in terms of convergent series with easily computable components. Second, it seems that the approximate solution in Example 5.1 using the decomposition method converges faster than the approximate solution using the variational iteration method while the approximate solution in Examples 5.2 and 5.3 using variational iteration method converges faster than the approximate solution using the decomposition method to the accurate solution. So, the accuracy of the methods depends on the nonlinear fractional differential equation and they can be used as an alternative methods for solving fractional partial differential equations. Third, the variational iteration method handles nonlinear equations without any need for the so-called Adomian polynomials. However, Adomain decomposition method provides the components of the exact solution, where these components should follow the summation given in (4.3).

Finally, the recent appearance of fractional differential equations as models in some fields of applied mathematics makes it necessary to investigate methods of solution for such equations (analytical and numerical) and we hope that this work is a step in this direction.

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