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Numerical approximations and Padé approximants for a fractional population growth model

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Abstract

This paper presents an efficient numerical algorithm for approximate solutions of a fractional population growth model in a closed system. The time-fractional derivative is considered in the Caputo sense. The algorithm is based on Adomian's decomposition approach and the solutions are calculated in the form of a convergent series with easily computable components. Then the Padé approximants are effectively used in the analysis to capture the essential behavior of the population $u(t)$ of identical individuals.

Keywords: Population dynamics; Fractional derivative; Adomian decomposition method; Padé approximants; Volterra integral equation

1. Introduction

Recently, the fractional derivative has drawn much attention due to its wide application in engineering, for example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [1], and the fluid-dynamic traffic model with fractional derivatives [2] can eliminate the deficiency arising from the assumption of continuum traffic flow. Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested in Ref. [3], and differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena [4]. A review of some applications of fractional derivatives in continuum and statistical mechanics is given by Mainardi [5]. This paper outlines reliable numerical strategies for solving the fractional population growth model of a species within a closed system. The model is characterized by the nonlinear fractional Volterra integro-differential equation

$$\frac{d^{\alpha}u}{dt^{\alpha}} = au - bu^2 - cu(t)\int_0^t u(s)ds, \quad u(0) = \beta,\ 0 < \alpha \leqslant 1, \tag{1.1}$$

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where $u = u(t)$ is the population of identical individuals at time t which exhibits crowding and sensitivity to the amount of toxins produced [6], α is a parameter describing the order of the time-fractional derivative, $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient, and $c > 0$ is the toxicity coefficient. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long run. If $c = 0$ we have the well-know logistic equation [6,7]. The last term contains the integral that indicates the "total metabolism" or total amount of toxins produced since time zero. The individual death rate is proportional to this integral, and so the population death rate due to toxicity must include a factor u. Since the system is closed, the presence of the toxic term always causes the population level to fall to zero in the long run, as will be seen later. The relative size of the sensitivity to toxins, c, determines the manner in which the population evolves before its extinction. The time-fractional derivative in Eq. (1.1) is considered in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 1$, the fractional equation reduces to a classical logistic growth model.

Several analytical and numerical methods have been proposed to solve the classical population growth model (1.1), when $\alpha = 1$. In [7], the successive approximations method was suggested to handle the population model (1.1), but was not implemented. In [6], singular perturbation methods were used to find a closed form approximations to the solutions of Eq. (1.1). In [6] the author showed that if c/ab is large, where the populations are strongly sensitive to toxins, that the solution is proportional to $\text{sech}^2(t)$. In this case the solution $u(t)$ has a smaller amplitude. Furthermore, for c/ab is small, where populations are weakly sensitive to toxins, the author showed that a rapid rise occurs along the logistic curve that will reach a peak and then followed by a slow exponential decay. In [8], the author used the series solution method and the Adomian decomposition method to present analytic approximation for the classical population growth model in a closed system. More recently, the Adomian decomposition method and Sinc method were applied independently in [9] to obtain approximate solution to the model. For more details about these investigations, the reader is advised to see Refs. [6,7,10,8,9] and the references therein.

In this paper, the Adomian decomposition method (ADM) [11,12] will be effectively used to approach (1.1). The Adomian algorithm introduces the solution in the form of a rapidly convergent series with elegantly computable terms. Furthermore, the behavior of the model in that it increases rapidly in the logistic curve and it decreases exponentially to extinction in the long run can be formally determined by using the Padé approximants of the series obtained. In addition, the effect of varying the order of the time-fractional derivative on the behavior of the population $u(t)$ for small c/ab and for large c/ab will be addressed by using the Padé approximants.

The paper is organized as follows. We begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results. In Section 3 we extend application of the decomposition method to construct our numerical solutions for fractional population growth model. Applications and numerical results are introduced in Section 4.

2. Basic definitions

We give some basic definitions and properties of the fractional calculus theory which are used in the following sections.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(>\mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geqslant 0$, of a function $f \in C_\mu$, $\mu \geqslant -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad \alpha > 0,\ x > 0,$$
$$J^0 f(x) = f(x).$$

Properties of the operator J^α can be found in [13], we mention only the following:

For $f \in C_\mu$, $\mu \geqslant -1$, $\alpha, \beta \geqslant 0$ and $\Gamma > -1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann–Liouville derivative have certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce now a modified fractional differential operator D^α proposed by Caputo in his work on the theory of viscoelasticity [14].

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) \mathrm{d}t, \tag{2.1}$$

for $m - 1 < \alpha \leqslant m$, $m \in \mathbb{N}$, $x > 0$, $f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 2.1. *If* $m - 1 < \alpha \leqslant m$, $m \in N$ *and* $f \in C_\mu^m$, $\mu \geqslant -1$, *then*

$$D^\alpha J^\alpha f(x) = f(x)$$

and

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

In this paper, the fractional derivatives are considered in the Caputo sense. The reason for adopting the Caputo definition is as follows [15]: to solve differential equations (both classical and fractional), we need to specify additional conditions in order to produce a unique solution. For the case of Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are akin to those of classical differential equations, and are therefore familiar to us. In contrast, for Riemann–Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives (and/or integrals) of the unknown solution at the initial point $x = 0$, which are functions of x. These initial conditions are not physical; furthermore, it is not clear how such quantities are to be measured from experiment, so that they can be appropriately assigned in an analysis. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann–Liouville and Caputo types see [16].

3. Decomposition method

The decomposition method requires that the fractional integro-differential equation (1.1) be expressed in terms of operator form as

$$D^\alpha u(t) = au(t) - bu^2(t) - c \int_0^t u(t)u(x) \mathrm{d}x, \tag{3.1}$$

where the fractional differential operator D^α is defined as in Eq. (2.1) denoted by

$$D^\alpha = \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}.$$

Applying the operator J^α, the inverse of the operator D^α, to both sides of Eq. (3.1) and using the initial condition lead to

$$u(t) = \beta + J^\alpha \left[au(t) - bu^2(t) - c \int_0^t u(t)u(x) \mathrm{d}x \right]. \tag{3.2}$$

The decomposition technique consist of representing the solution (3.2) as series

$$u(t)=\sum_{n=0}^{\infty}u_n(t), \tag{3.3}$$

where the components $u_n(t)$, $n\geqslant 0$ will be determined recursively. Substituting (3.3) into both sides of (3.2) gives

$$\sum_{n=0}^{\infty}u_n(t)=\beta+J^{\alpha}\left[a\sum_{k=0}^{\infty}u_n(t)-b\sum_{k=0}^{\infty}A_k(t)-c\sum_{k=0}^{\infty}\int_0^t B_k(x,t)\mathrm{d}x\right], \tag{3.4}$$

where the nonlinear terms $u^2(t)$ and $u(x)u(t)$ are represented by the so-called Adomian polynomials $A_n(t)$ and $B_n(s,t)$, respectively. i.e., we set

$$u^2(t)=\sum_{n=0}^{\infty}A_n(t), \tag{3.5}$$

$$u(x)u(t)=\sum_{n=0}^{\infty}B_k(x,t). \tag{3.6}$$

The Adomian polynomials $A_n(t)$ and $B_n(x,t)$ can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [12]. The first few terms of the Adomian polynomials for the nonlinear function $u^2(t)$ are derived as follows:

$$\begin{aligned}
&A_0(t)=u_0^2(t),\\
&A_1(t)=2u_0(t)u_1(t),\\
&A_2(t)=u_1^2(t)+2u_0(t)u_2(t),\\
&A_3(t)=2u_3(t)u_0(t)+2u_1(t)u_2(t),\\
&\quad\vdots
\end{aligned}$$

and for $B_n(x,t)$, we find

$$\begin{aligned}
&B_0(x,t)=u_0(x)u_0(t),\\
&B_1(x,t)=u_0(x)u_1(t)+u_0(t)u_1(x),\\
&B_2(x,t)=u_0(x)u_2(t)+u_1(x)u_1(t)+u_2(x)u_0(t),\\
&B_3(x,t)=u_0(x)u_3(t)+u_1(x)u_2(t)+u_2(x)u_1(t)+u_3(x)u_0(t),\\
&\quad\vdots
\end{aligned}$$

and so on, the other polynomials can be constructed in a similar way. We assign the zeroth component by $u_0=u(0)=\beta$. The remaining components $u_n(t)$, $n\geqslant 1$, can be determined completely such that each term is computed by using the previous term. Since u_0 is known

$$\begin{aligned}
&u_1(t)=J^{\alpha}\left(au_0(t)-bA_0(t)-c\int_0^t B_0(x,t)\mathrm{d}x\right),\\
&u_2(t)=J^{\alpha}\left(au_1(t)-bA_1(t)-c\int_0^t B_1(x,t)\mathrm{d}x\right),\\
&u_3(t)=J^{\alpha}\left(au_2(t)-bA_2(t)-c\int_0^t B_2(x,t)\mathrm{d}x\right),\\
&\quad\vdots\\
&u_{n+1}(t)=J^{\alpha}\left(au_n(t)-bA_n(t)-c\int_0^t B_n(x,t)\mathrm{d}x\right).
\end{aligned} \tag{3.7}$$

Finally, we approximate the solution $u(t)$ by the truncated series

$$\phi_N(t)=\sum_{n=0}^{N-1}u_n(t)\quad\text{and}\quad\lim_{N\to\infty}\phi_N(t)=u(t). \tag{3.8}$$

However, in many cases the exact solution in a closed form may be obtained. Moreover, the decomposition series solutions are generally converge very rapidly. The convergence of the decomposition series have investigated by several authors. The theoretical treatment of convergence of the decomposition method has been considered in the literature [17–21]. They obtained some results about the speed of convergence of this method. Abbaoui and Cherruault [22] have proposed a new approach of convergence of the decomposition series. The authors have given a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM in [22].

4. Applications and numerical results

In this section we use the ADM to find an approximate solution of the fractional model Eq. (1.1) together with the initial condition $u(0) = 0.1$, which represents the population growth in a closed system. To calculate the terms of the decomposition series (3.3) for $u(t)$, we substitute the initial condition and the corresponding Adomian polynomials into (3.7) and using Mathematica consequently, we obtain

$$\begin{aligned} u_0(t) &= 0.1, \\ u_1(t) &= \frac{1}{\alpha(1+\alpha)\Gamma(\alpha)}((0.1a(1+\alpha) - 0.01b(1+\alpha))t^{\alpha} - 0.01ct^{1+\alpha}), \\ u_2(t) &= \frac{1}{\Gamma(1+2\alpha)}(0.1a^2 - 0.03ab + 0002b^2)t^{2\alpha} - \frac{1}{\Gamma(2+2\alpha)}(0.03ac - 0.004bc + 0.01ac\alpha)t^{1+2\alpha} \\ &\quad + \frac{1}{\Gamma(3+2\alpha)}(0.03c^2 + 0.001c^2\alpha)t^{2+2\alpha}, \\ &\vdots \end{aligned} \tag{4.1}$$

The general form of the approximation of $u(t)$ is given by

$$\begin{aligned} u(t) = 0.1 &+ \frac{1}{\alpha(1+\alpha)\Gamma(\alpha)}\left((0.1a - 0.01b)(1+\alpha)t^{\alpha} - 0.01ct^{1+\alpha}\right) \\ &+ \frac{1}{\Gamma(1+2\alpha)}\left(0.1a^2 - 0.03ab + 0.002b^2\right)t^{2\alpha} + \cdots \end{aligned} \tag{4.2}$$

Our aim is to study the mathematical behavior of the solution of a fractional population growth model as the order of the fractional derivative changes. In particular, we seek to study the rapid growth along the logistic curve that will reach a peak, then slow exponential decayed for different values of α. It was formally shown by [8] that this goal can be achieved by forming Padé approximants [23] which have the advantage of manipulating the polynomial approximation into a rational function to gain more information about $u(t)$. It is well-known that Padé approximants will converge on the entire real axis [24] if $u(t)$ is free of singularities on the real axis. It is of interest to be noted that Padé approximants give results with no greater error bound [25] than approximation by polynomials.

To consider the behavior of solution of different values of α, we will take advantage of the explicit formula (4.2) available for $0 < \alpha \leqslant 1$, and consider the following two special cases:

Case I: We will examine the classical population growth model for small κ and large κ, where $\kappa = c/ab$. Setting $\alpha = 1$ and $\kappa = 0.1$ in (4.2), we reproduce the approximate solution obtained in [8,9] given by

$$\begin{aligned} u(t) = 0.1 &+ 0.9t + 3.55t^2 + 6.31666667t^3 - 5.5375t^4 - 63.70916667t^5 - 156.0804167t^6 \\ &- 18.47323411t^7 + 1056.288569t^8 + O(t^9). \end{aligned} \tag{4.3}$$

The [4/4] Padé approximants gives

$$[4/4] = \frac{0.1 + 0.468793t + 0.924957t^2 + 0.923129t^3 + 0.400423t^4}{1 - 4.31207t + 12.5582t^2 - 13.8806t^3 + 10.8683t^4}. \tag{4.4}$$

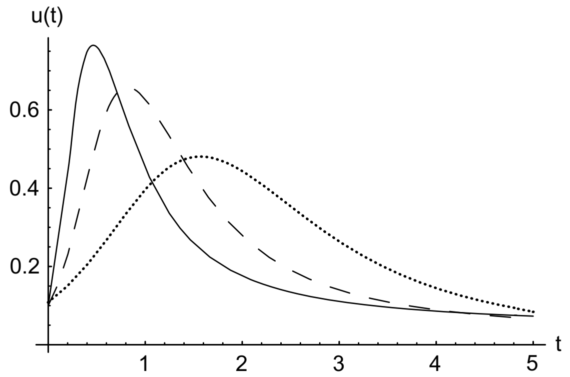

Fig. 1. [4/4] Padé approximants of $u(t)$ for $\alpha = 1$: (—) $\kappa = 0.1$, (---) $\kappa = 0.2$, (···) $\kappa = 0.5$.

Fig. 1 shows the [4/4] Padé approximants of $u(t)$ for $\kappa = 0.1$, 0.2, and 0.5. It can be seen from the figure that as κ increases, the amplitude of $u(t)$ decreases, whereas the exponential decay increases. These results are in full agreement with the results obtained in [8–10] using a phase-plane analysis.

Case II: In this case we will examine the fractional population growth model (1.1) when $\alpha = \frac{1}{2}$. Setting $\alpha = \frac{1}{2}$ and $\kappa = 0.1$ in (4.2) gives

$$u(t) = 0.1 + 1.01554t^{0.5} + 7.2t^{1} + 35.4964t^{1.5} + 90.422t^{2} - 321.158t^{2.5} - 5346.32t^{3} - 32307.8t^{3.5} - 82694.8t^{4} + O(t^{4.5}). \tag{4.5}$$

For simplicity, let $t^{1/2} = x$; then,

$$u(x) = 0.1 + 1.01554x^{1} + 7.2x^{2} + 35.4964x^{3} + 90.422x^{4} - 321.158x^{5} - 5346.32x^{6} - 32307.8x^{7} - 82694.8x^{8} + O(x^{9}). \tag{4.6}$$

Calculating the [6/6] Padé approximants and recalling that $x = t^{1/2}$, we get

$$[6/6] = \frac{0.1 + 0.755966\sqrt{t} + 7.72967t + 34.2007t^{3/2} + 409.984t^{2} + 570.184t^{5/2} + 2479.12t^{3}}{1 - 2.59575\sqrt{t} + 31.6576t - 147.559t^{3/2} + 3336.19t^{2} - 23232.9t^{5/2} + 89404.6t^{3}}. \tag{4.7}$$

Fig. 2 shows the [6/6] Padé approximants of $u(t)$ for $\kappa = 0.1$, and 0.5. The results in this figure show a rapid rise along the logistic curve and then a fast exponential decay to zero for small κ.

Fig. 3 shows the behavior of the solution of the fractional population growth model (1.1) for different values of α. The key finding of this graph is that when the order of the fractional derivative decreases, the amplitude of $u(t)$ decreases, whereas the exponential decay increases.

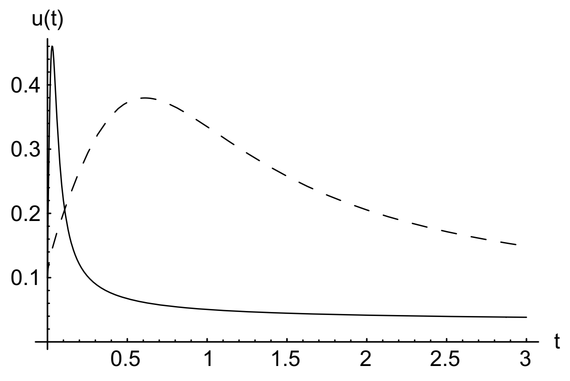

Fig. 2. [6/6] Padé approximants of $u(t)$ for $\alpha = 1/2$: (—) $\kappa = 0.1$, (---) $\kappa = 0.5$.

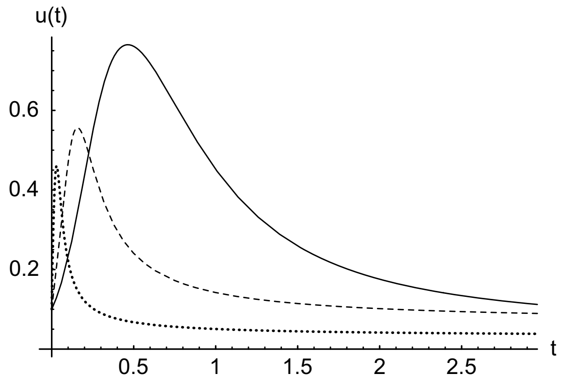

Fig. 3. Approximations of $u(t)$ for $\kappa = 0.1$: (—) $\alpha = 1.0$, (---) $\alpha = 3/4$, (···) $\alpha = 1/2$.

5. Conclusions

In this paper, the Adomian decomposition method has been successfully applied to finding the approximate solution of nonlinear fractional integro-differential equation (1.1). Analysis of the behavior of the model showed that it increases rapidly along the logistic curve followed by a slow exponential decay after reaching a maximum point, can be formally determined by using the Padé approximants. The fractional derivative is considered in the Caputo sense. Numerical results shows that when the order of the fractional derivative α decreases, the amplitude of $u(t)$ decreases, whereas the exponential decay increases.

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