

Non-standard finite difference schemes for solving fractional-order Rössler chaotic and hyperchaotic systems

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ABSTRACT

In this paper, the non-standard finite difference method (for short NSFD) is implemented to study the dynamic behaviors in the fractional-order Rössler chaotic and hyperchaotic systems. The Grünwald–Letnikov method is used to approximate the fractional derivatives. We found that the lowest value to have chaos in this system is 2.1 and hyperchaos exists in the fractional-order Rössler system of order as low as 3.8. Numerical results show that the NSFD approach is easy to implement and accurate when applied to differential equations of fractional order.

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1. Introduction

During the past decades fractional calculus has become a powerful tool to describe the dynamics of complex systems which appear frequently in several branches of science and engineering. Fractional differential equations, therefore find numerous applications in the field of visco-elasticity, control and robotics [1,2], feedback amplifiers, electrical circuits, electro-analytical chemistry, fractional multipoles, chemistry and biological sciences [3,4]. Recently, Magin [5] used fractional calculus to model some complex dynamics in biological tissues.

A great deal of effort has been expended over the past decade or so in attempting to find robust and stable numerical and analytical methods for solving fractional partial differential equations of physical interest. Numerical and analytical methods have included Adomian and variational methods [6,7]. Adams–Bashforth–Moulton method (ABMM) is one of the most used methods to solve fractional differential equations [8–13]. Cang et al. [14] presented the series solutions of nonlinear Riccati differential equations with fractional derivatives. Erjaee [15] investigated the saddle and Hopf bifurcation points of predator–prey fractional differential equations system with a constant rate harvesting using the non-standard finite difference method. Hussian et al. [16] used the non-standard discretization for solving ordinary differential equations of fractional order.

Chaotic systems have a profound effect on its numerical solutions and are highly sensitive to time step sizes. It will be beneficial to find a reliable analytical tool to test its long-term accuracy and efficiency. Also the hyperchaotic systems have more complex dynamical behaviors because it is defined as a chaotic system with two positive Lyapunov exponents. Li and Chen [17] have studied the dynamics of both the fractional-order generalizations of the well-known Rössler equation and the Rössler hyperchaos equation. The key finding of their study is that chaotic behavior exists in the fractional-order Rössler equation of order as low as 2.4, and hyperchaos exists in the fractional-order Rössler hyperchaos equation of order as low as 3.8. In this paper, we found that the lowest value to have chaos in this system is 2.1 and hyperchaos exists in the fractional-order Rössler system of order as low as 3.8.

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This paper is devoted to the construction of a non-standard discretization scheme given by Mickens to the Grünwald–Letnikov discretization process for solving the fractional-order Rössler chaotic and hyperchaotic systems. The NSFD by Mickens [18–23] has developed as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential, biological models and chaotic systems. The technique has many advantages over the classical techniques, and provides an efficient numerical solution.

2. Preliminaries and notations

In this section we give some basic definitions and properties of the Grünwald–Letnikov approximation and the non-standard finite difference method which are used further in this paper.

2.1. Grünwald–Letnikov approximation

We will begin with the single fractional differential equation, see [16,27–30],

$$D^\alpha y(t) = f(t, y(t)), \quad T \geq t \geq 0 \quad \text{and} \quad y(t_0) = 0, \quad (0.1)$$

where $\alpha > 0$ and D^α denote the fractional derivative, defined by

$$D^\alpha y(t) = J^{n-\alpha} D^n x(t), \quad (0.2)$$

where $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$ and J^n in the n th-order Riemann–Liouville integral operator defined as

$$J^n y(t) = \frac{1}{\Gamma(n)} \int_0^t (t - \tau)^{n-1} y(\tau) d\tau, \quad t > 0. \quad (0.3)$$

To apply Mickens' scheme, we have chosen the Grünwald–Letnikov method approximation for the one-dimensional fractional derivative as follows:

$$D^\alpha y(t) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{j=0}^N (-1)^j \binom{\alpha}{j} y(t - jh), \quad (0.4)$$

where $N = t/h$ and $[t]$ denotes the integer part of t and h is the step size. Therefore Eq. (0.1) is discretized as

$$\sum_{j=0}^N c_j^\alpha y(t_{n-j}) = f(t, y(t)), \quad n = 1, 2, 3, \dots, \quad (0.5)$$

where $t_n = nh$ and c_j^α are the Grünwald–Letnikov coefficients defined as

$$c_j^\alpha = \left(1 - \frac{1 + \alpha}{j}\right) c_{j-1}^\alpha, \quad (0.6)$$

and

$$c_0^\alpha = h^{-\alpha}, \quad j = 1, 2, 3, \dots \quad (0.7)$$

2.2. Non-standard finite difference method

We seek to obtain the NSFD [23] solution for a system of differential equations of the form

$$y'_k = f(t, y_1, y_2, \dots, y_m), \quad k = 1, 2, \dots, m, \quad (0.8)$$

where $f(t, y_k(t))$ is the nonlinear term in the differential equation. Using the finite difference method we have

$$y'_1 = \frac{y_{1,k+1} - y_{1,k}}{\phi_k(h)}, \quad (0.9)$$

$$y'_2 = \frac{y_{2,k+1} - y_{2,k}}{\phi_k(h)}, \quad (0.10)$$

⋮

$$y'_m = \frac{y_{m,k+1} - y_{m,k}}{\phi_m(h)}, \quad (0.11)$$

where ϕ_k is a function of the step size $h = \Delta t$. The function ϕ_k has the following properties:

$$\phi_k(h) = h + o(h^2), \quad (0.12)$$

where $h \rightarrow 0$.

Examples of functions $\phi_k(h)$ that satisfy (0.12) are h , $\sin(h)$, $\sinh(h)$, $e^h - 1$, $\frac{1-e^{-\lambda h}}{\lambda}$.

Nonlinear terms can in general be replaced by nonlocal discrete representations, for example,

$$y^2 \approx y_k y_{k+1}, \quad (0.13)$$

$$y^3 \approx \left(\frac{y_{k+1} + y_{k-1}}{2} \right) y_k^2, \quad (0.14)$$

where $h = T/N$, $t_n = nh$, $n = 0, 1, \dots, N \in \mathbb{Z}^+$.

3. Application

In order to demonstrate the performance and efficiency of the non-standard finite difference method for solving nonlinear fractional-order equations, we have applied the method to two examples. In the first example, we consider the fractional-order Rössler chaotic system, while in the second example, we consider the fractional-order Rössler hyperchaotic system.

3.1. The fractional-order Rössler chaotic system

Consider a fractional-order generalization of the Rössler system [24]. In this system, the integer-order derivatives are replaced by fractional-order derivatives, as follows:

$$D^{\alpha_1} x(t) = -y - z, \quad (0.15)$$

$$D^{\alpha_2} y(t) = x + ay, \quad (0.16)$$

$$D^{\alpha_3} z(t) = b + z(x - c), \quad (0.17)$$

where (x, y, z) are the state variables and (a, b, c) are positive constants. We will fix the two parameters b and c at 0.2 and 10, respectively. This system is equivalent to the classical integer-order Rössler system when $\alpha_1 = \alpha_2 = \alpha_3 = 1$, which is chaotic when $a = 0.15$, and for the fractional case, the parameter a is allowed to be varied.

Applying Mickens' scheme by replacing the step size h by a function $\phi(h)$ and using the Grünwald–Letnikov discretization method, yields the following equations:

$$\sum_{j=0}^{n+1} c_j^{\alpha_1} x(t_{n+1-j}) = -y(t_n) - z(t_n), \quad (0.18)$$

$$\sum_{j=0}^{n+1} c_j^{\alpha_2} y(t_{n+1-j}) = x(t_{n+1}) + a(2y(t_n) - y(t_{n+1})), \quad (0.19)$$

$$\sum_{j=0}^{n+1} c_j^{\alpha_3} z(t_{n+1-j}) = b + 2x(t_{n+1})z(t_n) - x(t_{n+1})z(t_{n+1}) - cz(t_n). \quad (0.20)$$

Comparing Eqs. (0.18)–(0.20) with Eqs. (0.15)–(0.17), we note the following:

1. The linear terms on the right-hand side of (0.15) have the form

$$-y = -y(t_n), \quad (0.21)$$

$$-z = -z(t_n). \quad (0.22)$$

2. The linear terms on the right-hand side of (0.16) have the form

$$x = x(t_{n+1}), \quad (0.23)$$

$$y = 2y - y \rightarrow 2y(t_n) - y(t_{n+1}). \quad (0.24)$$

3. The linear and nonlinear terms on the right-hand side of (0.17) have the form

$$xz = 2xz - xz \rightarrow 2x(t_{n+1})z(t_n) - x(t_{n+1})z(t_{n+1}), \quad (0.25)$$

$$-z = -z(t_n). \quad (0.26)$$

Doing some algebraic manipulation to Eqs. (0.18)–(0.20) yields the following relations

$$x(t_{n+1}) = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_1} x(t_{n+1-j}) - y(t_n) - z(t_n)}{c_0^{\alpha_1}}, \tag{0.27}$$

$$y(t_{n+1}) = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_2} y(t_{n+1-j}) + x(t_{n+1}) + 2ay(t_n)}{c_0^{\alpha_2} + a}, \tag{0.28}$$

$$z(t_{n+1}) = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_3} z(t_{n+1-j}) + b + 2x(t_{n+1})z(t_n) - cz(t_n)}{c_0^{\alpha_3} + x(t_{n+1})}, \tag{0.29}$$

where

$$c_0^{\alpha_1} = \phi(h)^{-\alpha_1}, \quad \phi(h) = \sin(h). \tag{0.30}$$

3.2. The fractional-order Rössler hyperchaotic system

The fractional-order Rössler hyperchaotic system [25] is given by

$$D^{\alpha_1} x(t) = -y - z, \tag{0.31}$$

$$D^{\alpha_2} y(t) = x + ay + w, \tag{0.32}$$

$$D^{\alpha_3} z(t) = b + xz, \tag{0.33}$$

$$D^{\alpha_4} w(t) = -cz + dw, \tag{0.34}$$

where (x, y, z, w) are the state variables and (a, b, c, d) are positive constants. In the case of $\alpha_i = 1 (i = 1, 2, 3, 4)$, the fractional system reduces to the classical Rössler hyperchaotic system, and it exhibits a hyperchaotic behavior when $a = 0.25, b = 3, c = 0.5$ and $d = 0.05$.

Following the same procedure as the previous example, we obtain the following non-standard finite difference scheme based on Grünwald–Letnikov discretization for the fractional-order Rössler hyperchaotic system

$$x(t_{n+1}) = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_1} x(t_{n+1-j}) - y(t_n) - z(t_n)}{c_0^{\alpha_1}}, \tag{0.35}$$

$$y(t_{n+1}) = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_2} y(t_{n+1-j}) + x(t_{n+1}) + ay(t_n) - w(t_n)}{c_0^{\alpha_2}}, \tag{0.36}$$

$$z(t_{n+1}) = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_3} z(t_{n+1-j}) + b + x(t_{n+1})z(t_n)}{c_0^{\alpha_3}}, \tag{0.37}$$

$$w(t_{n+1}) = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_4} w(t_{n+1-j}) - c + z(t_{n+1}) + dw(t_n)}{c_0^{\alpha_4}}, \tag{0.38}$$

where

$$c_0^{\alpha_1} = \phi_1(h)^{-\alpha_1}, \quad c_0^{\alpha_2} = \phi_2(h)^{-\alpha_2}, \tag{0.39}$$

$$c_0^{\alpha_3} = \phi_3(h)^{-\alpha_3}, \quad c_0^{\alpha_4} = \phi_4(h)^{-\alpha_4}, \tag{0.40}$$

and the nonlinearity xz in Eq. (0.33) is replaced by the nonlocal term $x(t_{n+1})z(t_n)$.

The functions $\phi_i (i = 1, 2, 3, 4)$ are chosen according to the non-diagonal elements of the Jacobian matrix of the original continuous system of the Rössler hyperchaotic system (0.31)–(0.34)

$$J_{ij} = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & a & 0 & 1 \\ z & 0 & x & 0 \\ 0 & 0 & -c & d \end{pmatrix}.$$

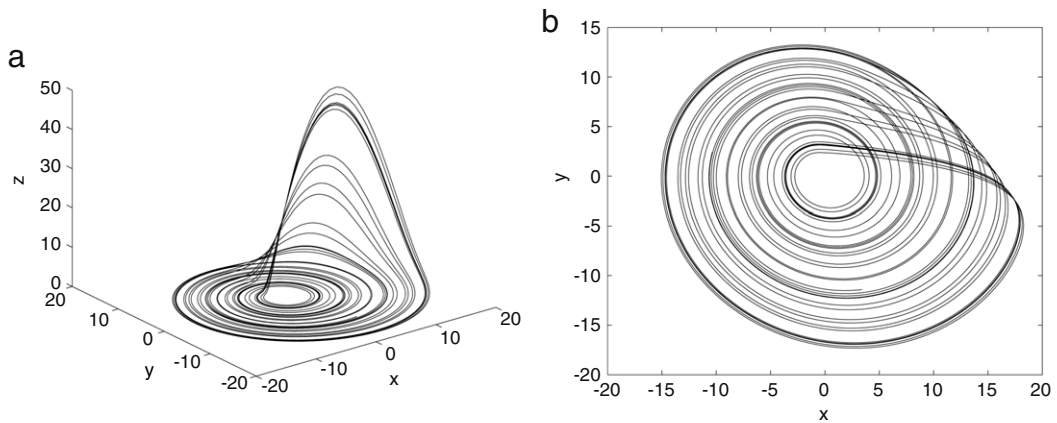


Fig. 1. Phase plot of chaotic attractor with $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $a = 0.15$: (a) in the $x - y - z$ space (b) in the $x - y$ space.

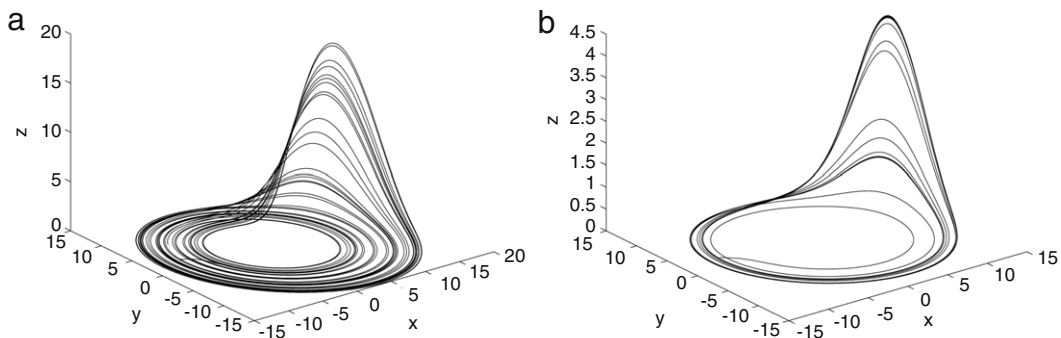


Fig. 2. Phase plot of chaotic attractor in the $x - y - z$ space: (a) $\alpha_1 = \alpha_2 = \alpha_3 = 0.9$, $a = 0.4$; (b) $\alpha_1 = 0.9$, $\alpha_2 = 0.8$, $\alpha_3 = 0.7$ and $a = 0.5$.

Since $J_{11} = 0$, we choose $\phi_1(h) = h$ [26]. The others are

$$\phi_2(h) = \frac{1 - e^{-ah}}{a}, \tag{0.41}$$

$$\phi_3(h) = \frac{1 - e^{-x_\phi h}}{x_\phi}, \tag{0.42}$$

$$\phi_4(h) = \frac{1 - e^{-dh}}{d}, \tag{0.43}$$

where x_ϕ is a fixed point of the Rössler hyperchaotic system (0.31)–(0.34) as follows

$$x_\phi = \frac{\sqrt{b}(c - ad)}{\sqrt{d}(c - ad)}. \tag{0.44}$$

4. Results and discussion

The fractional-order Rössler chaotic and hyperchaotic systems were numerically integrated using the non-standard finite difference scheme. The non-standard finite difference scheme is coded in the computer algebra package Matlab. The Matlab environment variable digits controlling the number of significant digits is set to 20 in all the calculations done in this paper. The time range studied in this work is $[0, 100]$ and we removed the first 500 points.

We consider the case $\alpha_1 = \alpha_2 = \alpha_3 = 1$ which corresponds to the classical Rössler system. Fig. 1 represents the phase portrait for chaotic solutions. The effective dimension \sum of Eqs. (0.18)–(0.20) is defined as the sum of orders $\alpha_1 + \alpha_2 + \alpha_3 = \sum$. We can see that the chaotic attractors of the fractional-order system are similar to that of the integer-order Rössler attractor as shown in Fig. 2.

Fig. 3 shows that the lowest order we found to yield chaos in this system is 2.1. From the graphical results in Figs. 2 and 3, it is concluded that the approximate solutions obtained using Mickens' non-standard discretization method is in good agreement with the approximate solutions obtained in [17].

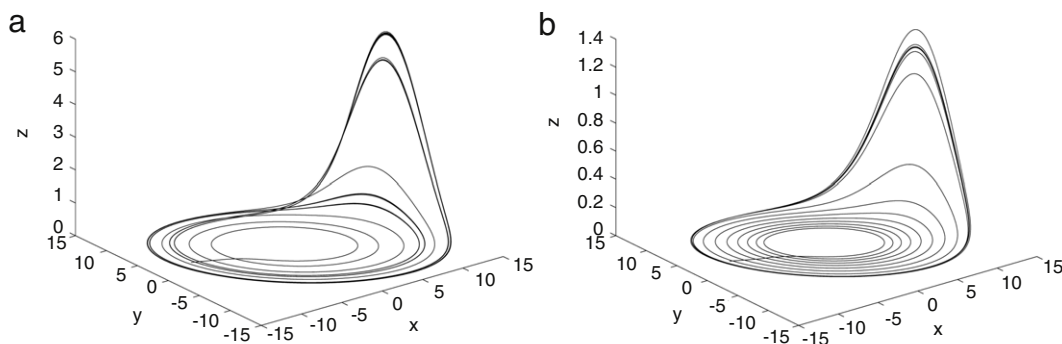


Fig. 3. Phase plot of chaotic attractor in the $x - y - z$ space: (a) $\alpha_1 = \alpha_2 = \alpha_3 = 0.75$, $a = 0.8$; (b) $\alpha_1 = \alpha_2 = \alpha_3 = 0.7$, $a = 0.91$.

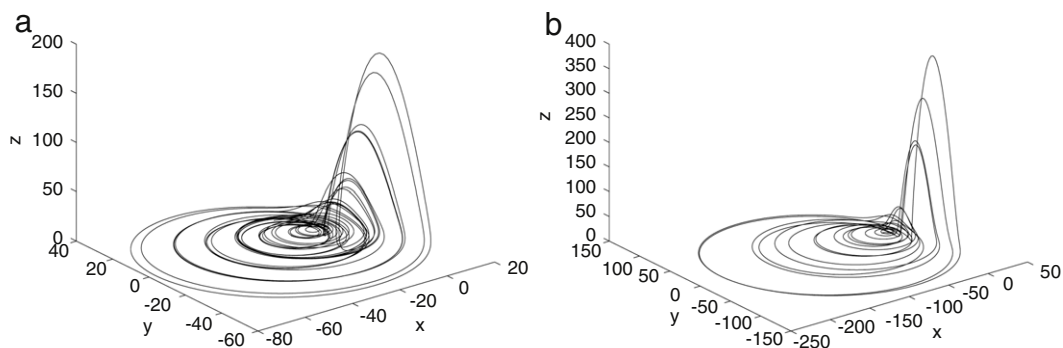


Fig. 4. Phase plot of hyperchaotic attractor in the $x - y - z$ space, (a) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, $a = 0.25$ (b) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.95$, $a = 0.32$.

Simulations were performed for the classical integer-order Rössler hyperchaotic system. This system has a hyperchaotic attractor when $a = 0.25$. The phase plot in the $x - y - z$ space is shown in Fig. 4(a). Fig. 4(b) shows that hyperchaos exists in the fractional-order Rössler hyperchaotic system with order as low as 3.8.

5. Conclusions

In this paper, a non-standard finite difference scheme given by Mickens has been successfully applied to find the numerical solutions of the fractional-order Rössler chaotic and hyperchaotic systems. The results indicated that the non-standard constructions are appropriate schemes because of the threshold and chaotic instabilities observed. However, the results obtained are in excellent agreement with those obtained by Li and Chen [17].

The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. Many of the results obtained in this paper can be extended to significantly more general classes of linear and nonlinear differential equations of fractional order.

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