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Inhibition or enhancement of chaotic convection via inclined magnetic field

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ABSTRACT

In this paper, we investigate the onset of convection in a horizontal layer of fluid which is heated from the underside. An inclined magnetic field is applied to the layer. The Galerkin truncated approximations were used to obtain a Lorenz-like model. The nonlinear system was solved by the fourth-order Runge–Kutta method. The results show that the Hartmann number and the angle of inclination of the magnetic field could inhibit or enhance the onset of chaotic convection.

1. Introduction

Thermal convection in a fluid layer is a fundamental paradigm for nonlinear dynamics including instabilities and bifurcations, pattern formation, chaotic dynamics and turbulence. Convection is said to be chaotic if nearby fluid elements typically diverge from each other exponentially in time. A truncated Galerkin expansion can be used to derive a system similar to the famous Lorenz equation [1–3] describing the dynamics.

A sudden transition to chaotic convection from steady for a low Prandtl number case was reported by Vadasz and Olek [4]. The transition is generated by a subcritical Hopf bifurcation which results in a solitary limit cycle. Vadasz [5] explained via local analytical results the appearance of this solitary limit cycle. Vadasz [6,7] employed similar approach for a convection problem in a pure fluid. Vadasz and Olek [8] found transition to chaos via a period doubling sequence of bifurcations for the moderate Prandtl number case. Sheu [9] showed that the fluid–solid interphase heat transfer could affect the route to chaos convection in a porous medium. In the moderate interphase heat transfer and small/moderate porosity-modified conductivity ratio case, a sudden transition to chaos was predicted. Furthermore, chaos occurred through period-doubling route for the weak interphase heat transfer and small porosity-modified conductivity ratio case. The effect of internal heat generation on the onset of chaotic convection in a porous medium for a low prandtl number case was investigated by Jawdat and Hashim [10]. It was found that a uniform internal heat generation could enhance the onset of chaotic convection. The inhibition of chaotic convection in nanofluids was shown possible by Jawdat et al. [11].

The study of magnetic field effects has important applications in physics and engineering. Sankar et al. [12] studied the effect of the directions of magnetic field radial or axial on the buoyancy-driven convection in a vertical cylindrical annulus filled with a low prandtl number electrically-conducting fluid. They found that the convection flow can be suppressed and

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the flow oscillations can be eliminated by the magnetic field, which is more effective when it is perpendicular to the direction of primary flow. Bednarz et al. [13] demonstrated a successful enhancement of convection with the application of a strong magnetic field generated by a super-conducting magnet. They also emphasized that using a strong magnetic field can suppress or invert the usual gravitational convection. The steady free convection in a rectangular cavity filled with a porous medium saturated with an electrically-conducting fluid under the influence of magnetic field was studied by Grosan et al. [14]. They found that the convective heat transfer can be reduced under the effect of the magnetic field, and the horizontal direction of the applied magnetic field is more effective in suppressing the convection flow than the vertical direction for a stronger magnetic field. Nadeem and Akram [15] investigated the peristaltic flow of a Williamson fluid in an inclined symmetric or asymmetric channel under the influence of an inclined magnetic field. They observed that the increase in Hartmann number and the volume flow rate will decrease the velocity profile, while it decreases with an increase in the inclination angle. Further, Nadeem and Akram [16] discussed the effects of partial slip on the peristaltic flow of a MHD Newtonian fluid in an asymmetric channel. They found that the temperature field and the pressure gradient decrease with the increase in slip parameter L, and magnetic field M, while the pressure rise increases in the peristaltic pumping region. a number of researches have discussed the peristaltic flow problems with the influence of inclined magnetic field in Newtonian and non-Newtonian fluids [17–19]. Sathiyamoorthy and Chamkha [20] considered steady, laminar, two-dimensional MHD natural convection within a liquid gallium filled square enclosure in the presence of an inclined magnetic field for different thermal boundary conditions. It was observed that the application of the magnetic field reduces the convective heat transfer rate in the cavity for any angle inclination. Idris and Hashim [21] also showed that delaying convective motion can be made possible via a magnetic field in a porous medium for a low Prandtl number case. Meanwhile, Mahmud and Hashim [22] showed that a constant, vertical magnetic field could suppress or enhance the chaotic convection.

The present work is aimed at extending the work of Mahmud and Hashim [22] to consider the influence of an inclined magnetic field on chaotic convection for a low Prandtl number case. Applying the truncated Galerkin approximation, an autonomous system is obtained and then analysed to study the effects of an inclined magnetic field on the transition to chaos.

2. Problem formulation

A schematic diagram of the problem geometry is shown in Fig. 1. The fluid layer is heated uniformly from below and cooled from above. In addition, the layer is subject to an externally-imposed inclined magnetic field of strength B with angle ϕ. Let x denote the spatial coordinate in the horizontal direction and z be the vertical axis pointing upwards such that $\hat{e_g} = -\hat{e}_z$.

A linear relationship between density and temperature is assumed and can be presented as $\rho = \rho_0[1 - \beta_*(T_* - T_c)]$, where β_* represents the thermal expansion coefficient. Subject to the Boussinesq approximation, the governing equations for an incompressible Newtonian fluid are the continuity equation, the suitably modified form of the Navier–Stokes equation and heat equation respectively,

$$\nabla \cdot V_* = 0, \tag{1}$$

$$\rho_0\left[\frac{\partial V_*}{\partial t_*} + V_* \cdot \nabla V_*\right] = -\nabla p_* + \nu_*\nabla^2 V_* + \rho\vec{g} + J \times B, \tag{2}$$

$$\frac{\partial T}{\partial t_*} + V_* \cdot \nabla T = \alpha_*\nabla^2 T, \tag{3}$$

$$\nabla \cdot J = 0; \quad J = \sigma(-\nabla\varphi + V_* \times B). \tag{4}$$

Here V_* denotes the velocity, t_* is time, T temperature, p_* pressure, ν_* fluid viscosity, α_* thermal diffusivity, J electric current density, φ electric potential, σ electric conductivity and ρ_0 is a reference value of density.

Garandet et al. [23] suggested that the electric potential in Eq. (4) was significantly reduced to $\nabla^2\varphi = 0$ for a 2-D steady-state situation. Since $\partial\varphi/\partial n = 0$, the unique solution is $\nabla\varphi = 0$. This means that the electric field vanishes everywhere.

The following transformations will non-dimensionalize Eqs. (1)–(4):

$$V = \frac{H_*}{\alpha_*}V_*, \quad p = \frac{H_*^2}{\rho_0\alpha_*^2}p_*, \quad \hat{t} = \frac{\alpha_*}{H_*^2}t_*,$$
$$T\Delta T_c = T_* - T_c, \quad x = \frac{x_*}{H_*}, \quad y = \frac{y_*}{H_*}, \quad z = \frac{z_*}{H_*}, \tag{5}$$

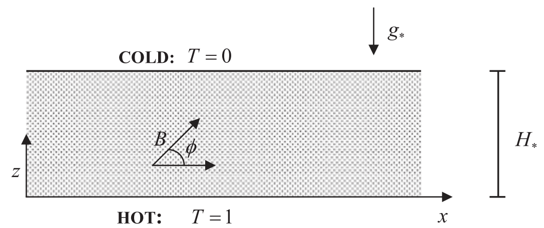


Fig. 1. A schematic representation of the horizontal fluid layer.

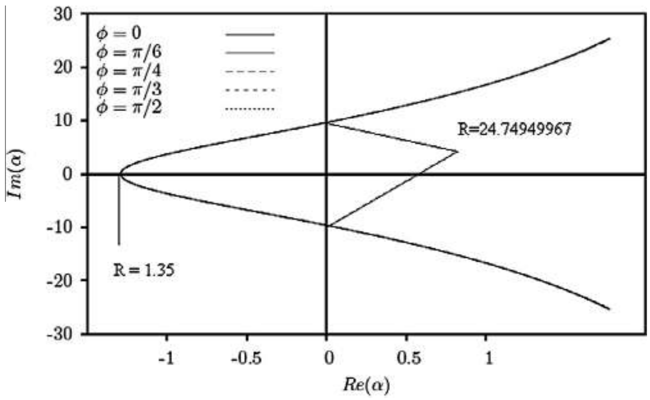


Fig. 2. Evolution of the complex eigenvalues with increasing Rayleigh number, for $Pr = 10,\ \lambda = 8/3,\ Ha = 0.5,\ \phi = 0,\ \pi/6,\ \pi/4,\ \pi/3,\ \pi/2$, with values of R for $\phi = \pi/4$.

Table 1
Values of R_{c1} and R when α_2, α_3 become equal and complex conjugates and when the loss of stability occurred for $Ha = 0.5$ and for different values of ϕ.

ϕ	R_{c1}	R (α_2, α_3 equal) (complex conjugate)	R (loss of stability occurred)
0	1.005628955	1.355	24.88482078
$\pi/6$	1.000142160	1.35	24.74056770
$\pi/4$	1.000482888	1.35	24.74949967
$\pi/3$	1.002956637	1.35	24.81445113
$\pi/2$	1.011257909	1.365	25.03373797

where $T_* - T_c$ and $\Delta T_c = T_H - T_c$ are the temperature variations and the characteristic temperature difference respectively.

Stress-free horizontal boundaries (i.e. no tangential shear stress) are considered for the present problem. Therefore, we have $V \cdot \hat{e}_n = 0$, where $\hat{e}_n$ is a unit vector normal to the boundary, and $\partial u/\partial z = \partial v/\partial z = \partial^2 w/\partial z^2 = 0$. The conditions for the temperature on the lower and upper boundaries are $T = 1$ and $T = 0$ respectively.

3. Solution of the problem and discussion

Taking the stream function $u = -\partial\psi/\partial z$ and $w = \partial\psi/\partial x$ and applying the curl on Eq. (2) gives,

$$\left[\frac{1}{Pr}\left(\frac{\partial}{\partial \hat{t}} - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial z}\right) - \nabla^2\right]\nabla^2\psi = Ra\left(\frac{\partial T}{\partial x}\right) - (Ha)^2\left[\frac{\partial^2\psi}{\partial x^2}\cos^2\phi + 2\frac{\partial^2\psi}{\partial x\partial z}\cos\phi\sin\phi + \frac{\partial^2\psi}{\partial z^2}\sin^2\phi\right], \tag{6}$$

$$\frac{\partial T}{\partial \hat{t}} - \frac{\partial\psi}{\partial z}\frac{\partial T}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}, \tag{7}$$

where the Prandtl number, $Pr = \nu_*/\alpha_*$, the Rayleigh number, $Ra = \beta_*\Delta T_c g_* H_*^3/\alpha_*\nu_*$, and the Hartmann number, $Ha = B_m(\sigma/\nu_*)^{1/2}$, B_m is the magnitude of the magnetic field B. On the boundaries we have $\psi = \partial\psi/\partial z = 0$.

To solve (6) and (7), we take equivalent Galerkin expansions of the stream function, ψ, and temperature, T, in the forms:

$$\psi = A_{11}\sin(\kappa x)\sin(\pi z), \tag{8}$$

$$T = 1 - z + B_{11}\cos(\kappa x)\sin(\pi z) + B_{02}\sin(2\pi z), \tag{9}$$

where A_{11}, B_{11} and B_{02} are amplitudes. The convective fixed points are of the form:

$$X = \frac{\widetilde{A}_{11}}{\sqrt{\lambda\frac{(R-G)}{G}}}, \quad Y = \frac{\widetilde{B}_{11}}{G\sqrt{\lambda\frac{(R-G)}{G}}}, \quad Z = \frac{-\widetilde{B}_{02}}{(R-G)}. \tag{10}$$

Now rescaling the time and amplitudes with respect to (10) results in the following Lorentz-like system:

$$\dot{X} = GPr(Y - X), \tag{11}$$

$$\dot{Y} = \left(\frac{R}{G}\right)X - Y - \left(\frac{R-G}{G}\right)XZ, \tag{12}$$

$$\dot{Z} = \lambda(XY - Z), \tag{13}$$

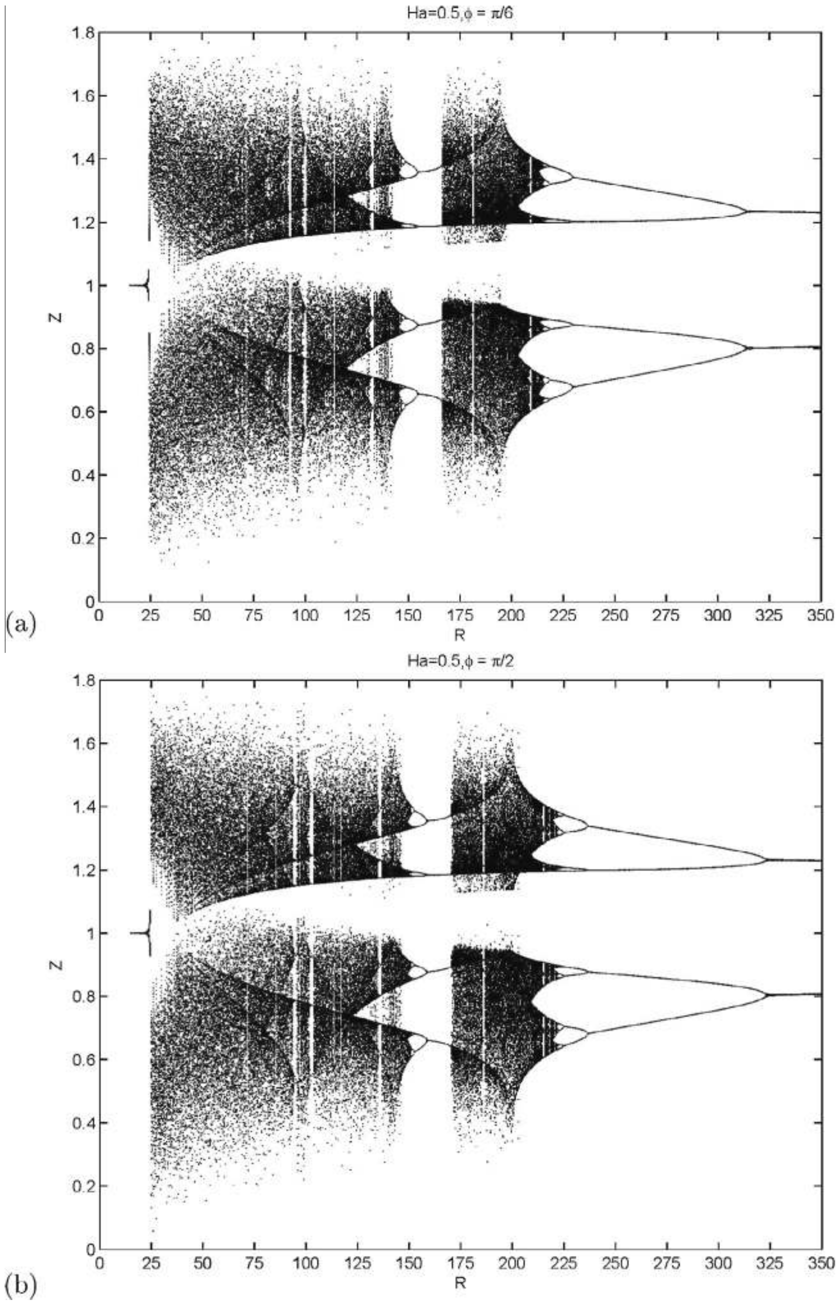


Fig. 3. Bifurcation diagrams of Z versus R representing the maxima and minima of the post-transient solution of $Z(t)$ for $Pr = 10, \lambda = 8/3, Ha = 0.5$ and $\phi = \pi/6, \pi/2$.

where

$$R = \frac{Ra}{Ra_{cr}}, \quad Ra_{cr} = \frac{(\kappa^2 + \pi^2)^3}{\kappa^2}, \quad \lambda = \frac{8}{[(\kappa/\kappa_{cr})^2 + 2]}, \tag{14}$$

$$G = 1 + \frac{4(Ha)^2}{9\pi^2}\left[\left(1/\sqrt{2}\right)\cos\phi - \sin\phi\right]^2 \tag{15}$$

and the primes $(')$ denote time derivatives $\mathrm{d}()/\mathrm{d}\tau$. Here κ is the wavenumber and the subscript cr denotes the critical point. Vadasz's system [7] is recovered when $Ha = 0$ (or $G = 1$). At the critical threshold, $\kappa = \kappa_{cr} = \pi/\sqrt{2}$, we have $\lambda = 8/3$.

The dynamics of system (11)–(13) about the fixed points corresponding to the motionless and convection solutions, respectively,

Table 2
Comparison between Vadasz's case ($Ha = 0$) and the inclined magnetic field case ($Ha = 0.5$) with various values of ϕ with $Pr = 10, \lambda = 8/3$ for R a solitary limit cycle signifying the loss of stability of the steady convection fixed points and the critical value of R at which the chaotic behaviour solution occurs.

	R (limit cycle)	R (critical value, chaotic behaviour)
Vadasz	23.474475752	24.737
($Ha = 0$)		
$\phi = 0$	24.567	24.88482078
$\phi = \pi/6$	24.4244	24.74056770
$\phi = \pi/4$	24.4332	24.74949967
$\phi = \pi/3$	24.4974	24.81445113
$\phi = \pi/2$	24.7142	25.03373797

$$X_1 = Y_1 = Z_1 = 0, \tag{16}$$
$$X_{2,3} = Y_{2,3} = \pm 1, \quad Z_{2,3} = 1, \tag{17}$$

will be determined via a stability analysis for the case $Pr = 10$ and $\lambda = 8/3$. For the fixed point (16), the characteristic polynomial determining the stability is

$$(-\lambda - \alpha)[(GPr + \alpha)(1 + \alpha) - PrR] = 0, \tag{18}$$

whose eigenvalues are

$$\alpha_1 = -\lambda = -8/3, \tag{19}$$
$$\alpha_{2,3} = \frac{1}{2}\left[-(1 + GPr) \pm \sqrt{(1 + GPr)^2 + 4Pr(R - G)}\right]. \tag{20}$$

Both α_1 and α_3 are negative and $\alpha_2 < 0$ when $R < G$, and hence the stability condition is

$$R_{c1} = R_{cr} = G, \tag{21}$$

which corresponds to $Ra_{cr} = (27\pi^4/4)G$. This means that there is a direct proportion between the Hartmann number Ha and the Rayleigh number Ra with fixed inclination angle ϕ.

The characteristic polynomial for the fixed points in (17) is

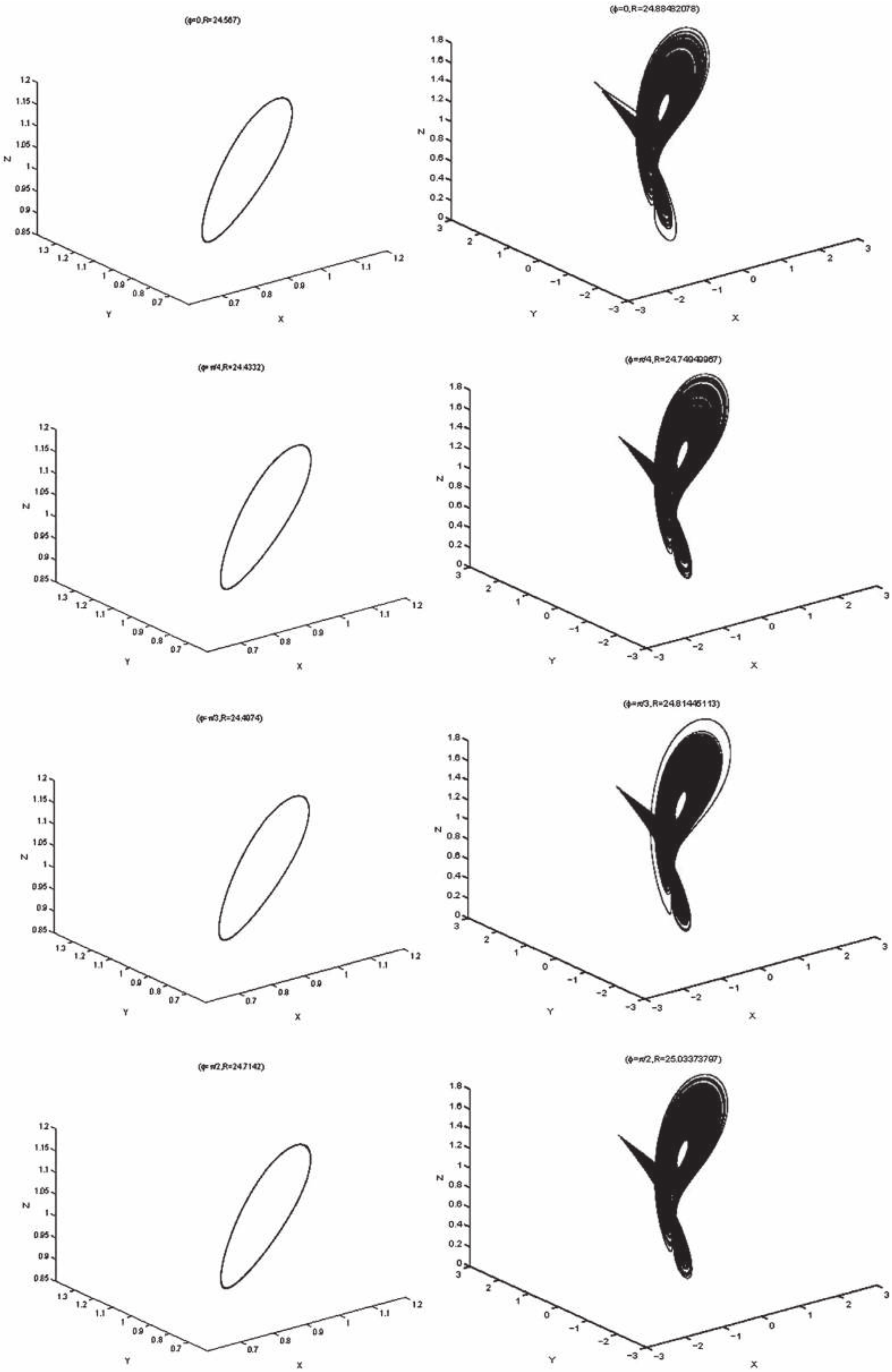
$$\alpha^3 + (1 + \lambda + GPr)\alpha^2 + (GPr\lambda + \lambda R/G)\alpha + 2Pr\lambda(R - G) = 0. \tag{22}$$

The first (smallest) eigenvalue α_1 is always real and negative over the entire range of parameter values. Thus, the stability of the convection fixed points depends on the other two roots. At a slightly supercritical value of R, these two roots are real and negative, and hence the fixed points are stable (simple nodes). As the value of R increase, these two roots travel along the real axis to the origin. Through this trip, they become equal then complex conjugate, and still have negative real parts. Hence, the fixed points are stable (spiral nodes). Increasing the value of R further, the real and imaginary parts of these two roots increase and extend over the imaginary axis. The convection fixed points lose their stability and hence periodic or chaotic solutions are dominated when the real part becomes nonnegative. This happens at a value of R given by

$$R_{c2} = \frac{G^2 Pr(3 + \lambda + GPr)}{GPr - \lambda - 1}, \tag{23}$$

which extends the corresponding result given in [7] to the inclined magnetic field case $Ha \neq 0$ (i.e. $G \neq 1$). For $Pr = 10, \lambda = 8/3$ and $Ha = 0.5$, the evolutions of the complex eigenvalues are presented in Fig. 2 with values of R for $\phi = \pi/4$. The values of R_{c1} and R at which α_2 and α_3 become equal and complex conjugate and the loss of stability occurred for $Ha = 0.5$ and for different values of ϕ are presented in Table 1.

MATLAB's built-in ODE45 with double precision and a stepsize of 0.001 was used to simulate system (11)–(13) for $0 \leqslant t \leqslant 210$. The following parameter values were fixed: $Pr = 10, \lambda = 8/3, Ha = 0.5$, and the initial values were taken as $X(0) = Y(0) = 0.8$ and $Z(0) = 0.92195$. Fig. 3 shows the bifurcation diagrams of Z versus R representing the maxima and minima of the post-transient solution of $Z(t)$ for $Pr = 10, \lambda = 8/3, Ha = 0.5$ and $\phi = \pi/6, \pi/2$. If we consider G as a function of ϕ with fixed Ha on the interval $[0, \pi/2]$, we notice that G decreases from $\phi = 0$ to ϕ with $\tan\phi = 1/\sqrt{2}$ and G increases from ϕ with $\tan\phi = 1/\sqrt{2}$ to $\phi = \pi/2$. Also, the critical value of the scaled Rayleigh number R decreases when G decreases and increases when G increases. Hence, the chaotic behaviour can be enhanced when G is decreasing and delayed when G is increasing. Table 2 shows the effect of ϕ on R for solitary limit cycle and onset of chaotic convection for the case $Ha = 0.5$. The corresponding results in terms of projection of trajectory data points on the X-Y-Z plane are presented in Fig. 4 which illustrates the transition from a limit cycle to chaos for the case $Ha = 0.5$ and several values of ϕ.



(a: $\phi = 0$)

(b: $\phi = \pi/4$)

(c: $\phi = \pi/3$)

(d: $\phi = \pi/2$)

Fig. 4. Computational results for the evolution of trajectories over time in the state space for two values of Rayleigh number (*R* where the solution is limit cycle and the critical value of *R*). The graphs represent the projection of the solution data points onto *X–Y–Z* plane for $Pr = 10,\ \lambda = 8/3,\ Ha = 0.5$ and $\phi = 0,\ \pi/4,\ \pi/3,\ \pi/2$.

4. Conclusions

In this work, the influence of an inclined magnetic field on the onset of chaotic convection in a horizontal layer of fluid heated from the underside has been studied in a low Prandtl number case. A direct proportion between the Hartmann number *Ha* and the Rayleigh number *Ra* is noticed with a fixed inclination angle ϕ. In comparison with Vadasz's case, the influence of inclined magnetic field can delay the onset of chaotic convection. Furthermore, the chaotic behaviour can be enhanced or delayed depending on the angle of inclination with a fixed Hartmann number. To sum up, the Hartmann number and the angle of inclination affect the transition from steady convection to chaos.

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