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Application of Legendre wavelets for solving fractional differential equations

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ABSTRACT

In this paper, we develop a framework to obtain approximate numerical solutions to ordinary differential equations (ODEs) involving fractional order derivatives using Legendre wavelet approximations. The properties of Legendre wavelets are first presented. These properties are then utilized to reduce the fractional ordinary differential equations (FODEs) to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique. Results show that this technique can solve the linear and nonlinear fractional ordinary differential equations with negligible error compared to the exact solution.

1. Introduction

Ordinary differential equations involving fractional order derivatives are used to model a variety of systems, of which an important engineering application lies in viscoelastic damping [1–3]. Another important application of fractional derivatives lies in control theory [4]. Surveys with a collection of applications in various fields can be found in [5,6].

In recent years, both mathematicians and physicists have devoted considerable effort to find robust and stable numerical and analytical methods for solving fractional differential equations of physical interest. Numerical and analytical methods have included the finite difference method [7–9], Adomian decomposition method [10–14], variational iteration method [15–18], homotopy perturbation method [19–22], generalized differential transform method [23–26], homotopy analysis method [27,28], and other methods [1,29].

The motivation of this paper is to extend the application of the Legendre wavelet approximations to solve linear and nonlinear differential equations of fractional order, by reducing the fractional ordinary differential equations (FODEs) to the solution of algebraic equations.

There are many different types of definitions of fractional calculus. For example, the Riemann–Liouville integral operator [5] of order α is defined by

$$(J^{\alpha}f)(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x \dfrac{f(t)}{(x-t)^{1-\alpha}} dt & \alpha > 0, t > 0, \\ f(x) & \alpha = 0. \end{cases} \quad (1)$$

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and its fractional derivative of order α ($\alpha \geq 0$) is normally used:

$$(\mathcal{D}_l^\alpha f)(x) = \left(\frac{d}{dx}\right)^m (J^{m-\alpha} f)(x), \quad (\alpha > 0, m-1 < \alpha < m), \tag{2}$$

where n is an integer. For Riemann–Liouville definition, one has

$$J^\alpha x^v = \frac{\Gamma(v+1)}{\Gamma(v+1+\alpha)} x^{v+\alpha}. \tag{3}$$

The Riemann–Liouville integral operator plays an important role in the development of the theory of fractional derivatives and integrals. However, it has some disadvantages for fractional differential equations with initial and boundary conditions. Therefore, we adopt here Caputo's definition [3,30], which is a modification of Riemann–Liouville definition:

$$(\mathcal{D}^\alpha f)(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt & (\alpha > 0, m-1 < \alpha < n,) \\ \dfrac{\partial^m f(x)}{\partial x^m} & \alpha = m, \end{cases} \tag{4}$$

where m is an integer. Caputo's integral operator has a useful property [3,30]:

$$(J^\alpha \mathcal{D}^\alpha f)(x) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)}{k!} x^k, \quad (x \geq 0, m-1 < \alpha < m,) \tag{5}$$

where m is an integer.

In this study, the fractional derivative is understood in the Caputo sense because of its applicability to real-world problems.

Wavelet theory is a relatively new and an emerging area in mathematical research. It has been applied in a wide range of engineering disciplines; particularly, wavelets are very successfully used in signal analysis for waveform representations and segmentations, time–frequency analysis and fast algorithms for easy implementation [31]. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [32].

In the present article, we are concerned with the application of Legendre wavelets to the numerical solution of FODEs. The method consists of conversion of ordinary differential equations involving fractional order to algebraic equations and expanding the solution by Legendre wavelets with unknown coefficients.

The article is organized as follows. In Section 2, we describe the basic formulation of wavelets and Legendre wavelets required for our subsequent development. Section 3 is devoted to the solution of FODEs using integral operator matrix and product operator matrix and Legendre wavelets. In Section 4, by considering numerical examples reported in our work, the accuracy of the proposed scheme is demonstrated.

2. The properties of Legendre wavelets

2.1. Wavelets and Legendre wavelets

Wavelets constitute a family of functions constructed from the dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as [33]:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right) \quad a, b \in R, a \neq 0. \tag{6}$$

If we restrict the parameters a and b to discrete values as $a = a_0^k$, $b = nb_0 a_0^k$, $a_0 > 1$, $b_0 > 0$ and n, and k positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \psi(a_0^k t - nb_0),$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{k,n}(t)$ forms an orthonormal basis [33].

Legendre wavelets $\psi_{nm}(t) = \psi(k, \hat{n}, m, t)$ have four arguments; $\hat{n} = 2n - 1$, $n = 1, 2, 3, \ldots, 2^{k-1}$, k can assume any positive integer, m is the order for Legendre polynomials and t is the normalized time. They are defined on the interval [0, 1) as follows:

$$\psi_{nm}(t) = \begin{cases} \sqrt{m+\dfrac{1}{2}}\, 2^{\frac{k}{2}} L_m(2^k t - \hat{n}), & \text{for } \dfrac{\hat{n}-1}{2^k} \leq t \leq \dfrac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

where $m = 0, 1, 2, \ldots, M-1$, $n = 1, 2, \ldots, 2^{k-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and the translation parameter $b = \hat{n}2^{-k}$. Here, $L_m(t)$ are the well-known Legendre polynomials of order m which are defined on the interval $[-1, 1]$, and can be determined with the aid of the following recurrence formulae:

$$L_0(t) = 1, \qquad L_1(t) = 1,$$
$$L_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) tL_m(t) - \left(\frac{m}{m+1}\right) L_{m-1}(t), \quad m = 1, 2, 3, \ldots.$$

The set $\{\psi_{nm}(t)\}$ is composed of an orthonormal system for $L^2\,[0, 1]$ [34].

2.2. Function approximation

A function $f(t)$ defined over $[0, 1)$ may be expanded by Legendre wavelet series as

$$f(t) = \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} c_{nm}\psi_{nm}(t), \tag{8}$$

where

$$c_{nm} = \langle f(t), \psi_{nm}(t)\rangle\,. \tag{9}$$

In (9), $\langle ., .\rangle$ denotes the inner product.

If the infinite series in Eq. (8) is truncated, then Eq. (8) can be written as

$$f(t) \cong \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t), \tag{10}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C(t) = \left[c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}\right]^T, \tag{11}$$

$$\Psi(t) = \left[\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1M-1}(t), \ldots, \psi_{2^{k-1}0}(t), \ldots, \psi_{2^{k-1}M-1}(t)\right]^T. \tag{12}$$

2.3. Legendre wavelets operational matrix of integration and product

The integration of the vector $\Psi(t)$ defined in (12) can be obtained as

$$\int_0^t \Psi(s)ds \cong P\Psi(t) \tag{13}$$

where P is a $2^{k-1}M \times 2^{k-1}M$ matrix given by [34] as

$$P = \frac{1}{2^k}\begin{bmatrix} L & F & \ldots & F \\ 0 & L & \ldots & F \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & L \end{bmatrix} \tag{14}$$

where

$$L = \begin{bmatrix} 1 & \frac{\sqrt{3}}{3} & 0 & \ldots & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \ldots & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ldots & \ddots & -\frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-5}} & \ddots & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & \ldots & \ldots & 0 & -\frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix}_{M\times M}$$

and

$$F = \begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{M \times M}.$$

The following property of the product of two Legendre wavelet vector functions will also be used:

$$\Psi(t)^T \Psi(t) C \approx \widehat{C} \Psi(t) \tag{15}$$

where C is a vector given in (11) and $\widehat{C}$ is a $2^{k-1}M \times 2^{k-1}M$ matrix, which is called the product operation of Legendre wavelet vector functions (see [34]). For $M = 3$, $k = 1$, the matrix $\widehat{C}$ is obtained:

$$\begin{bmatrix} c_{10} & c_{11} & c_{12} \\ c_{11} & c_{10} + \dfrac{2c_{12}}{\sqrt{5}} & \dfrac{2c_{11}}{\sqrt{5}} \\ c_{12} & \dfrac{2c_{11}}{\sqrt{5}} & c_{10} + \dfrac{2\sqrt{5}c_{12}}{7} \end{bmatrix}.$$

3. Applications to fractional ordinary differential equations

Consider the following initial value problem.

$$\mathcal{D}^\alpha x(t) + \mathcal{N}[x(t)] + \mathcal{L}[x(t)] = g(t) \quad \alpha > 0, \tag{16}$$

$$x^{(k)}(0) = c_k, \quad k = 0, 1, 2, \dots, m-1, m-1 < \alpha \le m \tag{17}$$

where $\mathcal{L}$ is a linear operator, $\mathcal{N}$ is a nonlinear operator and D^α is the Caputo fractional derivative of order α.

To solve Eqs. (16) and (17), we express Eq. (4) in the form

$$\mathcal{D}^\alpha x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{x^{(m)}(s)}{(t-s)^{\alpha-m+1}} ds,$$

and we use Legendre wavelets to approximate $x^{(m)}(t)$ as

$$x^{(m)}(t) = C^T \Psi(t), \tag{18}$$

then we have

$$x^{(k)}(t) = C^T P^k \Psi(t) + F_k^T \Psi(t), \qquad g(t) = G^T \Psi(t), \tag{19}$$

where coefficients of G, F_k are known and F_k can be obtained from the initial conditions and P is defined similarly to Eq. (14).

Since the basis of Legendre wavelets $\Psi(t)$ are polynomials, it is sufficient to calculate $\int_0^t \frac{s^n}{(t-s)^{\alpha-m+1}} ds$.

We have

$$\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{s^n}{(t-s)^{\alpha-m+1}} ds = \frac{t^{m+n-\alpha}\Gamma(n+1)}{\Gamma(m+n-\alpha+1)}. \tag{20}$$

In view of Eq. (18) we write Eq. (20) as

$$\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\Psi(s)}{(t-s)^{\alpha-m+1}} ds = S_1 \Psi(t), \tag{21}$$

where S_1 is the $2^{k-1} \times M$ matrix. Here, S_1 is an operational matrix of the fractional integration.

Now for the nonlinear part, by product operational matrix which is defined in (15), we have

$$\mathcal{N}[x(t)] = S_2 \Psi(t) \tag{22}$$

where S_2 is a vector function of the elements of the vector C^T.

For the linear part, we have

$$L[x(t)] = C^T S_3 \Psi(t) \tag{23}$$

where S_3 matrix is of order $2^{k-1} \times M$.

Substituting Eqs. (21)–(23) into Eq. (16), we find that:

$$C^T S_1 \Psi(t) + S_2 \Psi(t) + C^T S_3 \Psi(t) = G^T \Psi(t). \tag{24}$$

Simplifying $\psi(t)$ in Eq. (24), we get a nonlinear system in terms of C:

$$C^T S_1 + S_2 + C^T S_3 = G^T. \tag{25}$$

Solving this nonlinear system by mathematica software, we obtain the following solution of Eq. (16)

$$x(t) = C^T(P^m + F^T)\Psi(t).$$

4. Illustrative examples

To demonstrate the effectiveness of the method we consider here some fractional differential equations.

Example 1. Consider the Bagley–Torvik equation that governs the motion of a rigid plate immersed in a Newtonian fluid [1]

$$M\mathcal{D}^2x(t) + 2S\sqrt{\mu\rho}\mathcal{D}^{\frac{3}{2}}x(t) + Kx(t) = f(t) \quad x(0) = 1, x'(0) = 1. \tag{26}$$

This problem describes the motion of a large plate of the surface S and mass M in a Newtonian fluid with viscosity μ and density ρ. The plate is hanging on a massless spring of stiffness K. The function $f(t)$ represents the loading force.

In order to make comparison with the numerical solution in [35], we choose $M = 2S\sqrt{\mu\rho} = K = 1$ and $f(t) = t + 1$ in Eq. (26) to obtain

$$\mathcal{D}^2x(t) + \mathcal{D}^{\frac{3}{2}}x(t) + x(t) = 1 + t \quad x(0) = 1, x'(0) = 1. \tag{27}$$

We applied the Legendre wavelet approach to solve Eq. (27) with $M = 3$ and $k = 1$. In this case we obtain $G^T = \overrightarrow{0}$ and whereas the matrix $S_1 + S_2$ in this example is nonsingular matrix. Thus

$$C^T = \overrightarrow{0}.$$

Therefore, we have $x''(t) = 0$ and by the initial conditions we get

$$x(t) = 1 + t$$

which is the exact solution.

Example 2. Consider the composite fractional oscillation equation [29],

$$\mathcal{D}^\alpha x(t) + x(t) - t^2 - \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} = 0, \quad 0 < \alpha < 1, \tag{28}$$

with the initial condition $x(0) = 0$.

We applied the Legendre wavelet approach to solve Eq. (28) for different values of α. Using Section 3, we convert Eq. (28) to the following system

$$C^T(S_1 + P) = X^T. \tag{29}$$

If we set $\alpha = \frac{1}{2}$ and $M = 3$, $k = 1$, we obtain

$$t^2 + \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} \cong [0.935136, 0.735397, 0.138615].\Psi(t) = X^T\Psi(t).$$

Solving system (29), we have

$$x(t) = t^2,$$

which is the exact solution.

Example 3. Next, we consider the fractional differential equation [29]

$$\mathcal{D}^\alpha x(t) + x(t) = t^4 - \frac{1}{2}t^3 - \frac{3}{\Gamma(4-\alpha)}t^{3-\alpha} + \frac{24}{\Gamma(5-\alpha)}t^{4-\alpha}, \quad 0 < \alpha < 1, \tag{30}$$

with initial condition

$$x(0) = 0.$$

We solve this equation by Legendre wavelet for several values of M. Setting the Operational matrix of the fractional integration S_1 and matrix S_2 in Eq. (30), we have following system:

$$C^T(S_1 + P) = F^T, \tag{31}$$

where $F^T = F_1^T + F_2^T + F_3^T$.

For $M = 8, K = 1$ and $\alpha = \frac{1}{2}$, we have:

$$t^4 - \frac{1}{2}t^3 = \left[\frac{3}{40}, \frac{7}{40\sqrt{3}}, \frac{9}{56\sqrt{5}}, \frac{3}{40\sqrt{7}}, \frac{1}{210}, 0, 0, 0\right].\Psi(t) = F_1^T\Psi(t),$$

$$\frac{3}{\Gamma(4-\alpha)}t^{3-\alpha} \cong \left[\frac{16}{35\sqrt{\pi}}, \frac{16}{21\sqrt{3\pi}}, \frac{16\sqrt{\frac{5}{\pi}}}{231}, \frac{16}{429\sqrt{7\pi}}, -\frac{16}{15015\sqrt{\pi}}, \frac{16}{23205\sqrt{11\pi}}, -\frac{16}{74613\sqrt{13\pi}}, \frac{16\sqrt{\frac{5}{3\pi}}}{969969}\right] = F_2^T\Psi(t),$$

$$\frac{24}{\Gamma(5-\alpha)}t^{4-\alpha} \cong \left[\frac{256}{315\sqrt{\pi}}, \frac{256}{165\sqrt{3\pi}}, \frac{256\sqrt{\frac{5}{\pi}}}{1287}, \frac{256\sqrt{\frac{7}{\pi}}}{6435}, \frac{256}{36465\sqrt{\pi}}, -\frac{256}{188955\sqrt{11\pi}}, \frac{256}{1119195\sqrt{13\pi}}, -\frac{256\sqrt{\frac{5}{3\pi}}}{22309287}\right] = F_3^T\Psi(t).$$

By solving system (31), we obtain the solution of Eq. (30). In the case of $M = 8, K = 1$, we obtain

$$\alpha = \frac{1}{4} \Rightarrow x(t) = -1.45 \times 10^{-12} + 6.93 \times 10^{-11}t - 8.23 \times 10^{-10}t^2 - 0.5t^3 + t^4 + 1.37 \times 10^{-8}t^5 - 9.3 \times 10^{-9}t^6 + 2.49 \times 10^{-9}t^7$$

$$\alpha = \frac{1}{2} \Rightarrow x(t) = -1.81 \times 10^{-12} + 8.77 \times 10^{-11}t - 1.02 \times 10^{-9}t^2 - 0.5t^3 + t^4 + 1.54 \times 10^{-8}t^5 - 9.8 \times 10^{-9}t^6 + 2.49 \times 10^{-9}t^7$$

$$\alpha = \frac{3}{4} \Rightarrow x(t) = -8.44 \times 10^{-13} + 4.58 \times 10^{-11}t - 5.99 \times 10^{-10}t^2 - 0.5t^3 + t^4 + 1.151 \times 10^{-8}t^5 - 7.86 \times 10^{-9}t^6 + 2.12 \times 10^{-9}t^7.$$

Now if we increase M, for all α, our Legendre wavelet solution converges to

$$x(t) = t^4 - \frac{1}{2}t^3,$$

which is the exact solution.

Example 4. Consider the following fractional Riccati equation [36]:

$$\mathcal{D}^\alpha y(x) = -y(x)^2 + 1, \quad 0 < \alpha \le 1, \tag{32}$$

subject to initial condition

$$y(0) = 0.$$

The exact solution, when $\alpha = 1$, is

$$y(x) = \frac{e^{2x} - 1}{e^{2x} + 1}. \tag{33}$$

We applied the Legendre wavelet approach to solve Eq. (32) with $M = 25$, $k = 1$ and various values of α.

Figs. 1 and 2 show the approximate solutions for Example 4 obtained for different values of α using the Legendre wavelet scheme. From the graphical results in these figures, it is clear that the approximate solutions converge to the exact solution.

Example 5. Consider the following fractional Riccati equation [36]:

$$\mathcal{D}^\alpha y(t) = 2y(t) - y(t)^2 + 1 \quad 0 < \alpha \le 1 \tag{34}$$

subject to initial condition

$$y(0) = 0.$$

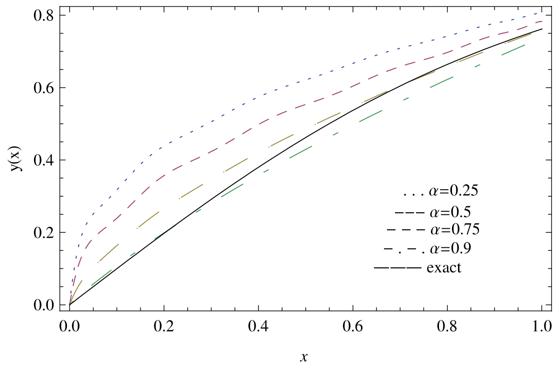

Fig. 1. Solutions of Eq. (32) using Legendre wavelets when $\alpha = 0.25, 0.5, 0.75, 0.9$.

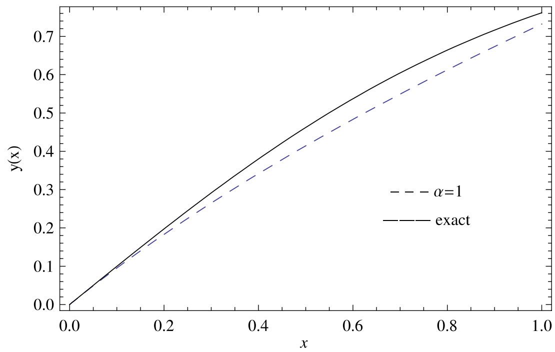

Fig. 2. Comparison between the Legendre wavelet solution and the exact solution when $\alpha = 1$.

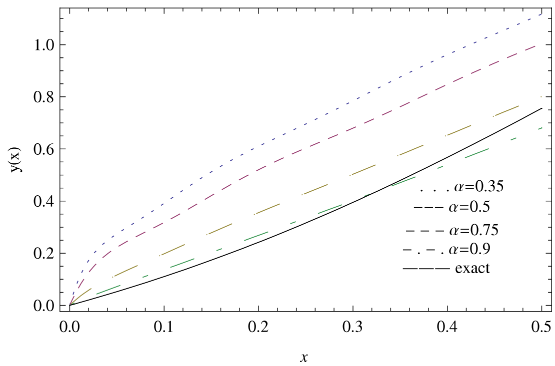

Fig. 3. Solutions of Eq. (34) using Legendre wavelets when $\alpha = 0.5, 0.75, 0.9$.

The exact solution, when $\alpha = 1$, is

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\mathrm{Log}\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \tag{35}$$

By setting $M = 25$ and $k = 1$, we obtain Legendre wavelet solution for various α. In Fig. 3, we show the Legendre wavelet solutions for various values of α. Fig. 4 show that the Legendre wavelet solution converges to the exact solution.

5. Conclusion

The aim of present work is to develop an efficient and accurate method for solving fractional differential equations. The problem has been reduced to solving a system of algebraic equations. Legendre wavelets are well behaved basic functions that are orthonormal on [0, 1]. Application of the wavelets allows the creation of more effective and faster algorithms than the ordinary ones. Illustrative examples are included to demonstrate the validity and applicability of the technique.

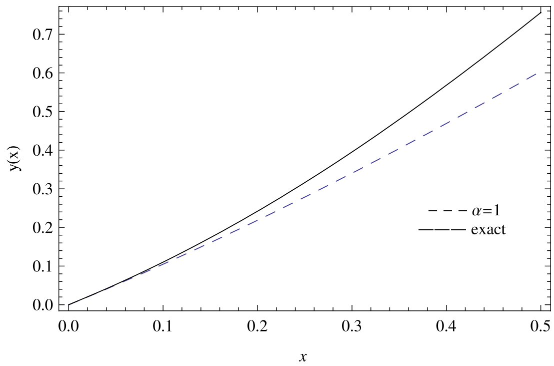

Fig. 4. Comparison of Legendre wavelet solution and the exact solution for $\alpha = 1$.

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