

Analytical approximate solutions of systems of fractional algebraic–differential equations by homotopy analysis method

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ABSTRACT

In this paper, we develop a framework to obtain approximate solutions to systems of algebraic–differential equations of fractional order by employing the homotopy analysis method (HAM). The study highlights the efficiency of the method and its dependence on the auxiliary parameter h . Numerical examples are examined to highlight the significant features of the HAM method. The method shows improvements over existing analytical techniques

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1. Introduction

Differential equations of fractional order have been found to be effective to describe some physical phenomena such as rheology, damping laws, fluid flow and so on [1–3]. A review of some applications of fractional calculus in continuum and statistical mechanics is given by Mainardi [1]. Recently, many important mathematical models can be expressed in terms of systems of algebraic–differential equations of fractional order. The solution of fractional differential equations is much involved. In general, there exists no method that yields exact solutions for fractional differential equations. Only approximate solutions can be derived using linearization or perturbation method. In recent years, much research has been focused on the numerical solution of systems of ordinary differential equations and algebraic–differential equations. Some numerical methods have been developed, such as implicit Runge–Kutta method [4], Pade approximation method [5–8], homotopy perturbation method [9–11], Adomain decomposition method [12–18] and variation iteration method [19–22]. Liao [23] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method. This method has been successfully applied to solve many types of nonlinear problems [24–37]. In this paper, we further apply the homotopy analysis method to solve systems of algebraic–differential equations of fractional order. By means of introducing an auxiliary parameter h , we can enlarge the convergence region of series solution. Furthermore, we give some examples to demonstrate the efficiency and effectiveness of the proposed method.

2. Basic definitions

For the concept of fractional derivative, we will adopt Caputo's definition which is a modification of the Riemann–Liouville definition.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ , $\mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x)$ is continuous in $[0, \infty)$. Clearly $C_\mu \subset C_\beta$ if $\beta < \mu$.

Definition 2.2. A function $f(x)$, $x > 0$, is said to be in the space C_μ^m , $m \in N \cup \{0\}$, if $f^{(m)} \in C_\mu$.

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Definition 2.3. The left sided Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$\begin{aligned} J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, x > 0, \\ J^0 f(x) &= f(x). \end{aligned} \quad (2.1)$$

Definition 2.4. Let $f \in C_{-1}^m$, $m \in N \cup \{0\}$ then the Caputo fractional derivative of $f(x)$ is defined as

$$D_*^\alpha f(x) = \begin{cases} [J^{m-\alpha} f^{(m)}(x)], & m-1 < \alpha < m, m \in N, \\ \frac{d^m f(x)}{dx^m}, & \alpha = m. \end{cases} \quad (2.2)$$

Hence, we have the following properties [1–3]

$$\begin{aligned} 1. \quad J^\alpha J^\nu f &= J^{\alpha+\nu} f, \quad \alpha, \nu \geq 0, f \in C_\mu, \mu \geq -1. \\ 2. \quad J^\alpha x^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\gamma+\alpha}, \quad \alpha > 0, \gamma > -1, x > 0. \\ 3. \quad J^\alpha D_*^\alpha f(x) &= f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0, m-1 < \alpha \leq m. \end{aligned} \quad (2.3)$$

3. Homotopy analysis method

In this section, we employ the homotopy analysis method [23] to the discussed problem. To show the basic idea, let us consider the following system of fractional algebraic–differential equations

$$\begin{aligned} D_*^{\alpha_i} x_i(t) &= f_i(t, x_1, \dots, x_n, x'_1, x'_2, \dots, x'_n), \quad i = 1, 2, 3, \dots, n-1, t \geq 0, 0 < \alpha_i \leq 1, \\ 0 &= g(t, x_1, \dots, x_n), \end{aligned} \quad (1)$$

subject to the initial conditions

$$x_i(0) = a_i, \quad i = 1, 2, 3, \dots, n. \quad (2)$$

By means of generalizing the traditional homotopy method, we can construct the so-called zero-order deformation equations

$$\begin{aligned} (1-q)\mathcal{L}_i[\phi_i(t, q) - x_{i0}(t)] &= q h_i H_i(t) [D_*^{\alpha_i} \phi_i(t, q) - f_i(t, \phi_1(t, q), \phi_2(t, q), \dots, \phi_n(t, q)), \\ &\quad \frac{d}{dt} \phi_1(t, q), \frac{d}{dt} \phi_2(t, q), \dots, \frac{d}{dt} \phi_n(t, q)], \quad i = 1, 2, \dots, n-1, \\ (1-q)[\phi_n(t, q) - x_{n0}(t)] &= -q h_n H_n(t) [g(t, \phi_1(t, q), \phi_2(t, q), \dots, \phi_n(t, q))], \end{aligned} \quad (3)$$

subject to the initial conditions

$$\phi_i(0, q) = a_i, \quad i = 1, 2, 3, \dots, n, \quad (4)$$

where $q \in [0, 1]$ is an embedding parameter, \mathcal{L}_i are auxiliary linear operators satisfying $\mathcal{L}_i(0) = 0$, $x_{i0}(t)$ are initial guesses satisfy the initial conditions (2), $h_i \neq 0$ are auxiliary parameters, $H_i(t)$ are auxiliary functions and $\phi_i(t, q)$ are unknown functions. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $q = 0$, and $q = 1$, we get

$$\phi_i(t, 0) = x_{i0}(t), \quad \phi_i(t, 1) = x_i(t), \quad i = 1, 2, 3, \dots, n, \quad (5)$$

respectively. Thus, as q increasing from 0 to 1, the solutions $\phi_i(t, q)$ varies from the initial guesses $x_{i0}(t)$ to the solutions $x_i(t)$. Expanding $\phi_i(t, q)$ in Taylor series with respect to q , one has

$$\phi_i(t, q) = x_{i0}(t) + \sum_{m=1}^{\infty} x_{im}(t) q^m, \quad i = 1, 2, 3, \dots, n, \quad (6)$$

where

$$x_{im}(t) = \frac{1}{m!} \left. \frac{\partial^m \phi_i(t, q)}{\partial q^m} \right|_{q=0}, \quad i = 1, 2, 3, \dots, n. \quad (7)$$

If the auxiliary linear operators \mathcal{L}_i , the auxiliary parameters h_i , the auxiliary functions $H_i(t)$ and the initial guesses $x_{i0}(t)$, are so properly chosen, then the series (6) converges at $q = 1$, one has

$$x_i(t) = x_{i0}(t) + \sum_{m=1}^{\infty} x_{im}(t), \quad i = 1, 2, 3, \dots, n. \quad (8)$$

Define the vectors

$$\vec{x}_{im} = \{x_{i0}(t), x_{i1}(t), x_{i2}(t), \dots, x_{im}(t)\}, \quad i = 1, 2, 3, \dots, n. \tag{9}$$

Differentiating Eq. (3) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$ we have the so-called m th-order deformation equations

$$\begin{aligned} \mathcal{L}_i[x_{im}(t) - \chi_m x_{i(m-1)}(t)] &= \hbar_i H_i(t) R_{im}(\vec{x}_{i(m-1)}(t)), \quad i = 1, 2, 3, \dots, n-1, \\ x_{nm}(t) &= \chi_m x_{n(m-1)}(t) + \hbar_n H_n(t) R_{nm}(\vec{x}_{n(m-1)}(t)), \end{aligned} \tag{10}$$

subject to the initial conditions

$$x_{im}(0) = 0, \quad i = 1, 2, 3, \dots, n, \quad m = 1, 2, 3, \dots, \tag{11}$$

where

$$\begin{aligned} R_{im}(\vec{x}_{i(m-1)}) &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} [D_*^{\alpha_i} \phi_i(t, q) - f_i(t, \phi_1(t, q), \phi_2(t, q), \dots, \phi_n(t, q)), \\ &\quad \frac{d}{dt} \phi_1(t, q), \frac{d}{dt} \phi_2(t, q), \dots, \frac{d}{dt} \phi_n(t, q)] \Big|_{q=0}, \quad i = 1, 2, 3, \dots, n-1, \\ R_{nm}(\vec{x}_{n(m-1)}) &= -\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} [g(t, \phi_1(t, q), \phi_2(t, q), \dots, \phi_n(t, q))] \Big|_{q=0}, \end{aligned} \tag{12}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{13}$$

Define \mathcal{L}_i^* to be an operators such that

$$\mathcal{L}_i^* \mathcal{L}_i[x_i(t)] = x_i(t) + K_i(t), \quad \mathcal{L}_i \mathcal{L}_i^* = I, \quad i = 1, 2, 3, \dots, n-1, \tag{14}$$

where I is the identity operator. For the special case, we put $H_i(t) = 1, i = 1, 2, \dots, n$, and so we have the following m th-order deformation equations

$$\begin{aligned} x_{im}(t) &= \chi_m x_{i(m-1)}(t) + \hbar_i \mathcal{L}_i^* [R_{im}(\vec{x}_{i(m-1)}(t))] + K_i(t), \quad i = 1, 2, 3, \dots, n-1, \\ x_{nm}(t) &= \chi_m x_{n(m-1)}(t) + \hbar_n R_{nm}(\vec{x}_{n(m-1)}(t)). \end{aligned} \tag{15}$$

4. Numerical results

To demonstrate the effectiveness of the method, we consider the following systems of fractional algebraic–differential equations

Example 4.1. Consider the following system of fractional algebraic–differential equations

$$\begin{aligned} D_*^\alpha x(t) - ty'(t) + x(t) - (1+t)y(t) &= 0, \quad 0 < \alpha \leq 1, \\ y(t) - \sin t &= 0, \end{aligned} \tag{16}$$

subject to the initial conditions

$$x(0) = 1, \quad y(0) = 0. \tag{17}$$

For the special case when $\alpha = 1$ the exact solution is [12]

$$x(t) = e^{-t} + t \sin t, \quad y(t) = \sin t. \tag{18}$$

We start with initial approximations

$$x_0(t) = 1, \quad y_0(t) = 0, \tag{19}$$

and linear operator

$$\mathcal{L} = D_*^\alpha, \quad \mathcal{L}^* = J^\alpha. \tag{20}$$

According to the formula (12), we can construct the homotopy as follows

$$\begin{aligned} R_{1m}(\vec{x}_{m-1}(t)) &= D_*^\alpha x_{m-1}(t) - ty'_{m-1}(t) + x_{m-1}(t) - (1+t)y_{m-1}(t), \\ R_{2m}(\vec{y}_{m-1}(t)) &= y_{m-1}(t) - (1 - \chi_m) \sin t, \end{aligned} \tag{21}$$

and the m th-order deformation equations for $m \geq 1$ become

$$\begin{aligned} x_m(t) &= \chi_m x_{m-1}(t) + \hbar_1 J^\alpha [R_{1m}(\vec{x}_{m-1}(t))], \\ y_m(t) &= \chi_m y_{m-1}(t) + \hbar_2 R_{2m}(\vec{y}_{m-1}(t)). \end{aligned} \tag{22}$$

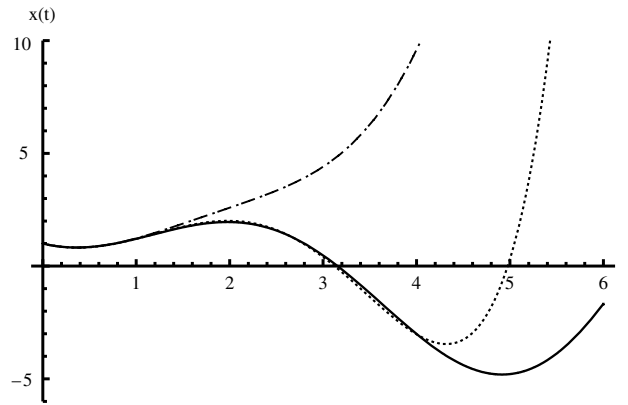


Fig. 1. Plots of solution of system (16) when $\alpha = 1$: Solid line: exact solution, dash-dotted line: $h_1 = h_2 = -1$, dotted line: $h_1 = -0.6, h_2 = -1$.

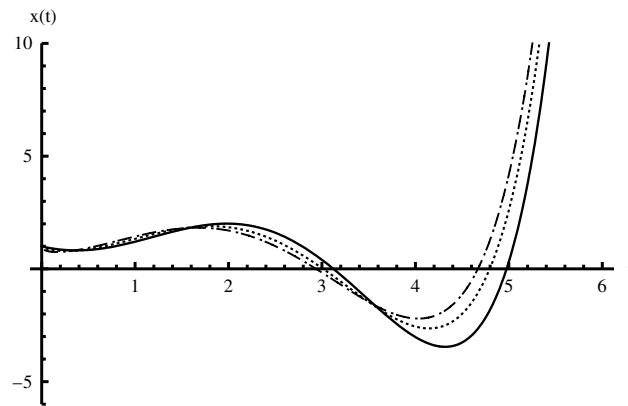


Fig. 2. Plots of solution of system (16) when $h_1 = -0.6, h_2 = -1$ Solid line: $\alpha = 1$, dotted line: $\alpha = 0.75$, dash-dotted line: $\alpha = 0.5$.

Using formula (22), we can obtain the following series solution

$$\begin{aligned}
 x(t) = & 1 + \frac{h_1^4 + 4h_1^3 + 6h_1^2 + 4h_1}{\Gamma(1 + \alpha)} t^\alpha + \frac{2h_1h_2(h_1^2 + h_2^2 + h_1h_2 + 4h_1 + 4h_2 + 6)}{\Gamma(2 + \alpha)} t^{\alpha+1} \\
 & + \left[\frac{h_1^2(3h_1^2 + 8h_1 + 3)}{\Gamma(1 + 2\alpha)} + \frac{3h_1^2 4^{-\alpha} \sqrt{\pi}}{\Gamma(1 + \alpha)\Gamma(\frac{1}{2} + \alpha)} \right] t^{2\alpha} \\
 & + \frac{2h_1h_2(h_1^2 + h_2^2 + h_1h_2 + 4h_1 + 4h_2 + 6)}{\Gamma(3 + \alpha)} t^{\alpha+2} + \frac{2h_1^2h_2(2h_1 + h_2 + 4)}{\Gamma(2 + 2\alpha)} t^{2\alpha+1} \\
 & + \frac{h_1^3(3h_1 + 4)}{\Gamma(1 + 3\alpha)} t^{3\alpha} - \frac{4h_1h_2(h_1 + h_2 + 3)}{\Gamma(4 + \alpha)} t^{\alpha+3} + \frac{2h_1^2h_2(2h_1 + h_2 + 4)}{\Gamma(3 + 2\alpha)} t^{2\alpha+2} \\
 & + \frac{2h_1^3h_2}{\Gamma(2 + 3\alpha)} t^{3\alpha+1} + \frac{h_1^4}{\Gamma(1 + 4\alpha)} t^{4\alpha} + \dots, \\
 y(t) = & -h_2(h_2^4 + 5h_2^3 + 10h_2^2 + 10h_2 + 5) \sin t.
 \end{aligned}
 \tag{23}$$

If we set $h_1 = h_2 = -1$, and $\alpha = 1$ in (23), then we obtain the same solutions given by Celik, Bayram and Yeloglu [12] using Adomian decomposition method. However, the results given by the Adomian decomposition method converge to the corresponding numerical solutions in a rather small region, as shown in Fig. 1. But different from this method, the homotopy analysis method gives a greater region of convergence with the exact solution, by choosing proper values for the auxiliary parameters h_1 and h_2 and by using a suitable auxiliary linear operator $\mathcal{L} = D_*^\alpha$. Fig. 2 shows the HAM approximate solutions for various values of α which have the same trajectories.

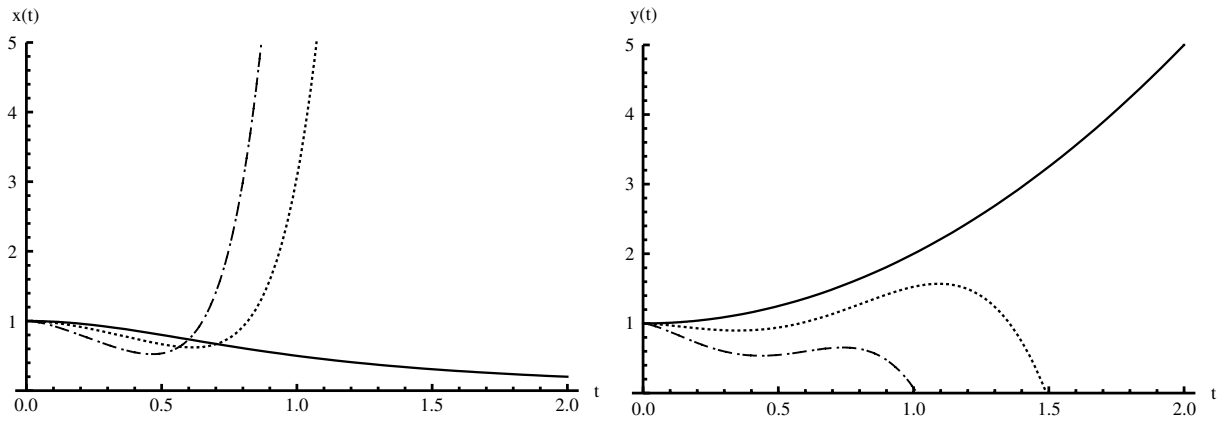


Fig. 3. Plots of solution of system (24) when $h_1 = h_2 = -1$ Solid line: $\alpha = 1$, dotted line: $\alpha = 0.75$, dash-dotted line: $\alpha = 0.5$, dashed line: $\alpha = 0.25$.

Example 4.2. Consider the following system of fractional algebraic–differential equations

$$\begin{aligned} D_*^\alpha x(t) &= -2tx^2(t) + x(t)y(t) - 1, \quad 0 < \alpha \leq 1, \\ y(t) &= x(t)y(t) + t^2, \end{aligned} \tag{24}$$

subject to the initial conditions

$$x(0) = y(0) = 1. \tag{25}$$

For the special case when $\alpha = 1$ the exact solution is

$$x(t) = \frac{1}{1+t^2}, \quad y(t) = t^2 + 1. \tag{26}$$

To solve the above problem using HAM, we choose the following initial approximations

$$x_0(t) = 1, \quad y_0(t) = 1, \tag{27}$$

and the linear operator (20). With reference to the formula (12) we can construct the homotopy as follows

$$\begin{aligned} R_{1m}(\tilde{x}_{m-1}(t)) &= D_*^\alpha x_{m-1}(t) + 2t \sum_{i=0}^{m-1} x_i(t)x_{m-1-i}(t) - \sum_{i=0}^{m-1} x_i(t)y_{m-1-i}(t) + (1 - \chi_m), \\ R_{2m}(\tilde{y}_{m-1}(t)) &= y_{m-1}(t) - \sum_{i=0}^{m-1} x_i(t)y_{m-1-i}(t) - (1 - \chi_m)t^2, \end{aligned} \tag{28}$$

and by using the m th-order deformation Eq. (22), we obtain the following series solution

$$\begin{aligned} x(t) &= 1 + \frac{2h_1(h_1+2)}{\Gamma(2+\alpha)}t^{\alpha+1} - \frac{2h_1^2}{\Gamma(2+2\alpha)}t^{2\alpha+1} + \frac{2h_1h_2}{\Gamma(3+\alpha)}t^{\alpha+2} + \frac{8h_1^2\Gamma(3+\alpha)}{\Gamma(2+\alpha)\Gamma(3+2\alpha)}t^{2\alpha+2} + \dots, \\ y(t) &= 1 - 2h_2t^2 - \frac{2h_1h_2}{\Gamma(2+\alpha)}t^{\alpha+1} + \dots, \end{aligned} \tag{29}$$

Setting $h_1 = h_2 = -1$, and $\alpha = 1$ in (29), then we obtain the following series solution

$$\begin{aligned} x(t) &= 1 - t^2 + t^4 - \dots = \sum_{k=0}^{\infty} (-1)^k t^{2k}, \\ y(t) &= 1 + t^2, \end{aligned} \tag{30}$$

which is the exact solution for system (24), this means that the best values for auxiliary parameters h_1, h_2 are $h_1 = h_2 = -1$. Fig. 3 shows the HAM approximate solutions for various values of α .

Example 4.3. Consider the following system of fractional algebraic–differential equations

$$\begin{aligned} D_*^{\alpha_1} x(t) - ty'(t) + t^2z'(t) + x(t) - (1+t)y(t) + (t^2+2t)z(t) &= 0, \\ D_*^{\alpha_2} y(t) - tz'(t) - y(t) + (t-1)z(t) &= 0, \quad 0 < \alpha_1, \alpha_2 \leq 1, \\ z(t) - \sin t &= 0, \end{aligned} \tag{31}$$

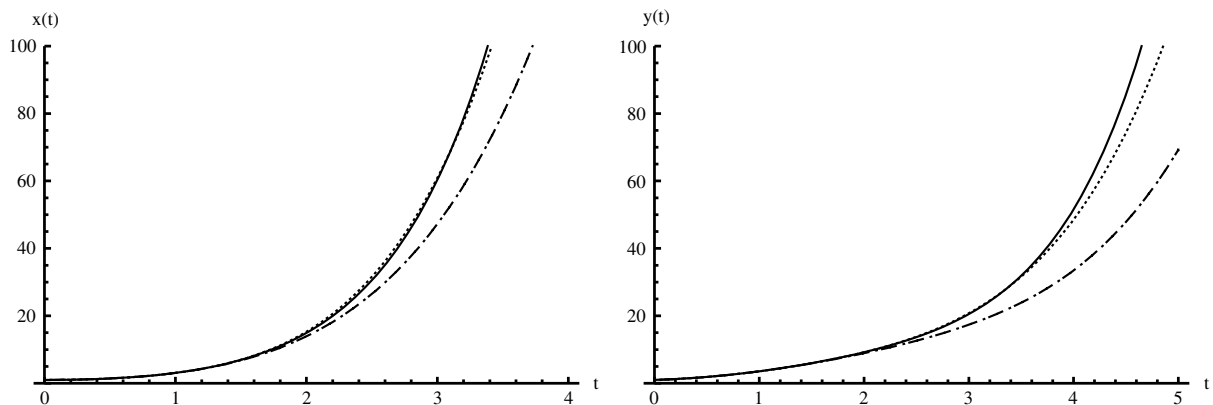


Fig. 4. Plots of solution of system (31) when $\alpha_1 = \alpha_2 = 1$ Solid line: exact solution, dash-dotted line: $h_1 = h_2 = h_3 = -1$, dotted line: $h_1 = -1, h_2 = -1.35, h_3 = -1$.

subject to the initial conditions

$$x(0) = 1, \quad y(0) = 1, \quad z(0) = 0. \tag{32}$$

When $\alpha_1 = \alpha_2 = 1$, the exact solution is [12]

$$x(t) = e^{-t} + te^t, \quad y(t) = e^t + t \sin t, \quad z(t) = \sin t. \tag{33}$$

To solve system (31) by means of homotopy analysis method, we start with initial approximations

$$x_0(t) = 1, \quad y_0(t) = 1, \quad z_0(t) = 0, \tag{34}$$

and linear operators

$$\mathcal{L} = D_*^{\alpha_i}, \quad \mathcal{L}_i^* = J_i^{\alpha_i}, \quad i = 1, 2. \tag{35}$$

In view of the formula (12) we can construct the homotopy as follows

$$\begin{aligned} R_{1m}(\bar{x}_{m-1}(t)) &= D_*^{\alpha_1} x_{m-1}(t) - ty'_{m-1}(t) + t^2 z'_{m-1}(t) + x_{m-1}(t) - (t + 1)y_{m-1}(t) + (t^2 + 2t)z_{m-1}(t), \\ R_{2m}(\bar{y}_{m-1}(t)) &= D_*^{\alpha_2} y_{m-1}(t) - tz'_{m-1}(t) - y_{m-1}(t) + (t - 1)z_{m-1}(t), \\ R_{3m}(\bar{z}_{m-1}(t)) &= z_{m-1}(t) - (1 - \chi_m) \sin t, \end{aligned} \tag{36}$$

and the m th-order deformation equations for $m \geq 1$ become

$$\begin{aligned} x_m(t) &= \chi_m x_{m-1}(t) + \hbar_1 J^{\alpha_1} [R_{1m}(\bar{x}_{m-1}(t))], \\ y_m(t) &= \chi_m y_{m-1}(t) + \hbar_2 J^{\alpha_2} [R_{2m}(\bar{y}_{m-1}(t))], \\ z_m(t) &= \chi_m z_{m-1}(t) + \hbar_3 R_{3m}(\bar{z}_{m-1}(t)). \end{aligned} \tag{37}$$

Using the formula (37), we can obtain the following series solution

$$\begin{aligned} x(t) &= 1 - \frac{\hbar_1(\hbar_1^2 + 3\hbar_1 + 3)}{\Gamma(2 + \alpha_1)} t^{\alpha_1 + 1} + \frac{\hbar_1 \hbar_2(\hbar_1 + \hbar_2 + 3)(1 + \alpha_2)}{\Gamma(1 + \alpha_1 + \alpha_2)} t^{\alpha_1 + \alpha_2} - \frac{6\hbar_1 \hbar_2(\hbar_1 + \hbar_3 + 1)}{\Gamma(3 + \alpha_1)} t^{\alpha_1 + 2} \\ &\quad - \frac{\hbar_1^2(2\hbar_1 + 3)}{\Gamma(2 + 2\alpha_1)} t^{2\alpha_1 + 1} + \frac{\hbar_1 \hbar_2}{\Gamma(2 + \alpha_1 + \alpha_2)} [(\hbar_1 + \hbar_2 + 3)(1 + \alpha_2) - 2(2 + \alpha_2)] t^{\alpha_1 + \alpha_2 + 1} \\ &\quad + \frac{\hbar_1^2 \hbar_2(1 + \alpha_2)}{\Gamma(1 + 2\alpha_1 + \alpha_2)} t^{2\alpha_1 + \alpha_2} - \frac{\hbar_1 \hbar_2^2(1 + 2\alpha_2)}{\Gamma(1 + \alpha_1 + 2\alpha_2)} t^{\alpha_1 + 2\alpha_2} + \dots, \\ y(t) &= 1 - \frac{\hbar_2(\hbar_2^2 + 3\hbar_2 + 3)}{\Gamma(1 + \alpha_2)} t^{\alpha_2} + \frac{2\hbar_2 \hbar_3(\hbar_2 + \hbar_3 + 3)}{\Gamma(2 + \alpha_2)} t^{\alpha_2 + 1} + \hbar_2^2 \left[\frac{(1 + 2\hbar_2)}{\Gamma(1 + 2\alpha_2)} \right. \\ &\quad \left. + \frac{2\sqrt{\pi}4^{-\alpha_2}}{\Gamma(1 + \alpha_2)\Gamma(\frac{1}{2} + \alpha_2)} \right] t^{2\alpha_2} - \frac{2\hbar_2 \hbar_3(\hbar_2 + \hbar_3 + 3)}{\Gamma(3 + \alpha_2)} t^{\alpha_2 + 2} - \frac{2\hbar_2^2 \hbar_3}{\Gamma(2 + 2\alpha_2)} t^{2\alpha_2 + 1} - \frac{\hbar_2^3}{\Gamma(1 + 3\alpha_2)} t^{3\alpha_2} + \dots, \\ z(t) &= -\hbar_3(\hbar_3^4 + 5\hbar_3^3 + 10\hbar_3^2 + 10\hbar_3 + 5) \sin t. \end{aligned} \tag{38}$$

Setting $\hbar_1 = \hbar_2 = \hbar_3 = -1$, and $\alpha_1 = \alpha_2 = 1$ in (38), then we obtain the same solutions given by Celik, Bayram and Yeloglu [12] using Adomian decomposition method. Similarly as in Example (1), Fig. 4 shows that choosing auxiliary

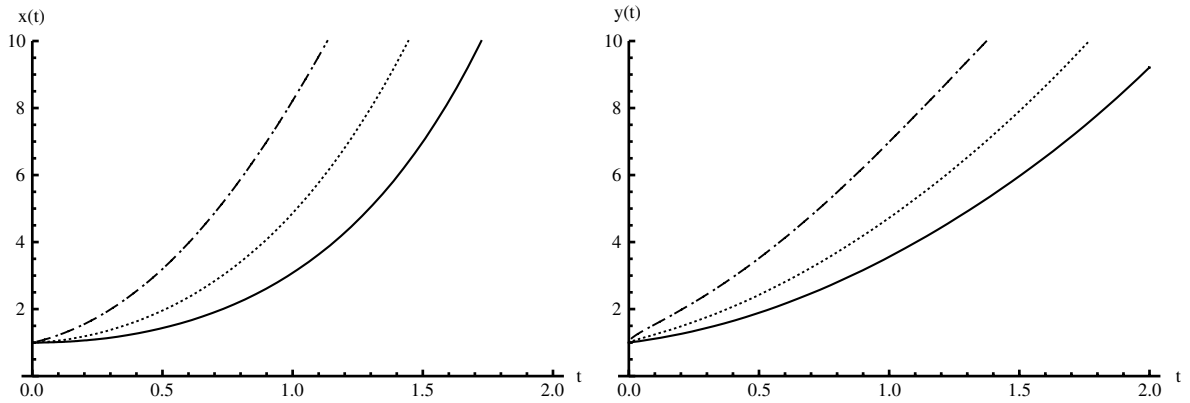


Fig. 5. Plots of solution of system (31) when $h_1 = -1, h_2 = -1.35, h_3 = -1$ Solid line: $\alpha_1 = \alpha_2 = 1$, dotted line: $\alpha_1 = \alpha_2 = 0.75$, dash-dotted line: $\alpha_1 = \alpha_2 = 0.5$.

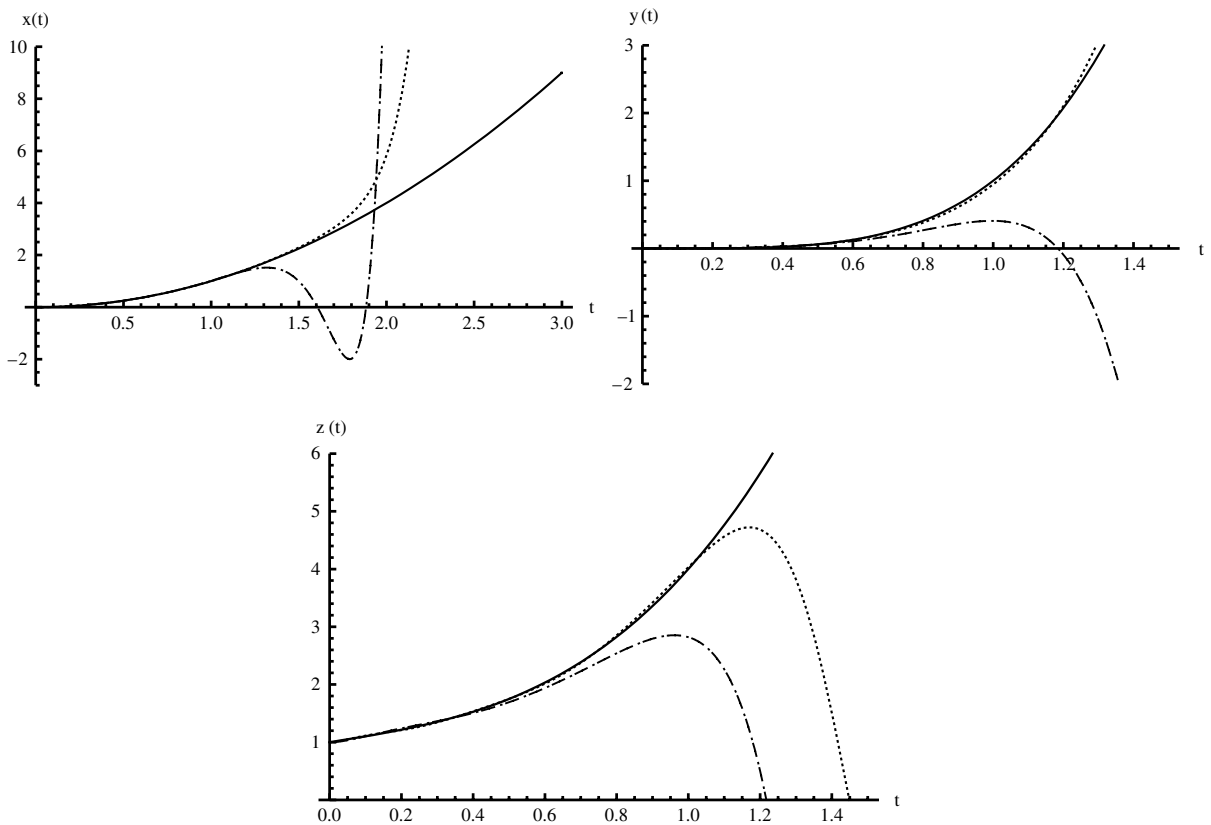


Fig. 6. Plots of solution of system (31) when $\alpha_1 = \alpha_2 = 1$. Solid line: exact solution, dash-dotted line: $h_1 = h_2 = h_3 = -1$, dotted line: $h_1 = -0.6, h_2 = -0.9, h_3 = -2$.

linear operator (35) and using the values of auxiliary parameters $h_1 = -1, h_2 = -1.35, h_3 = -1$, the HAM gives greater convergence with the exact solutions. Fig. 5 shows the HAM approximate solutions for various values of α_1, α_2 .

Example 4.4. Consider the following system of fractional algebraic–differential equations

$$\begin{aligned} D_*^{\alpha_1} x(t) - x^2(t) + y(t) - 2t &= 0, \\ D_*^{\alpha_2} y(t) - 2z(t) + 2(t + 1) &= 0, \quad 0 < \alpha_1, \alpha_2 \leq 1, \\ tz(t) + y(t) - x(t) - 3x^2(t) - t &= 0, \end{aligned} \tag{39}$$

subject to the initial conditions

$$x(0) = y(0) = 0, \quad z(0) = 1. \tag{40}$$

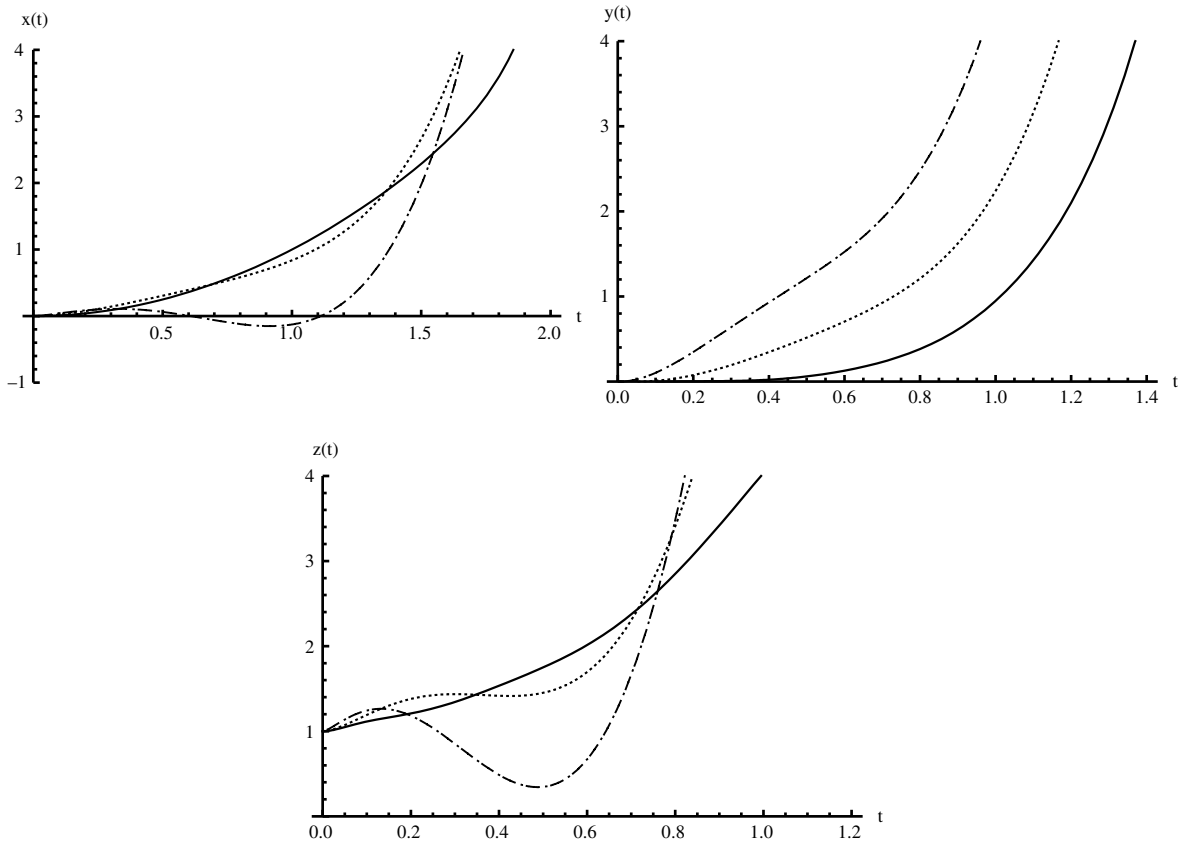


Fig. 7. Plots of solution of system (31) when $h_1 = -0.6$, $h_2 = -0.9$, $h_3 = -2$. Solid line: $\alpha_1 = \alpha_2 = 1$, dotted line: $\alpha_1 = \alpha_2 = 0.75$, dash-dotted line: $\alpha_1 = \alpha_2 = 0.5$.

For the special case when $\alpha_1 = \alpha_2 = 1$ the exact solution is

$$x(t) = t^2, \quad y(t) = t^4, \quad z(t) = 2t^3 + t + 1. \tag{41}$$

If we use the linear operator (35) and the following initial approximations

$$x_0(t) = 0, \quad y_0(t) = 0, \quad z_0(t) = 1, \tag{42}$$

and according to the formula (12) we can construct the homotopy as follows

$$\begin{aligned} R_{1m}(\vec{x}_{m-1}(t)) &= D_*^{\alpha_1} x_{m-1}(t) - \sum_{i=0}^{m-1} x_i(t)x_{m-1-i}(t) + y_{m-1}(t) - 2t(1 - \chi_m), \\ R_{2m}(\vec{y}_{m-1}(t)) &= D_*^{\alpha_2} y_{m-1}(t) - 2z_{m-1}(t) + 2(t + 1)(1 - \chi_m), \\ R_{3m}(\vec{z}_{m-1}(t)) &= tz_{m-1}(t) + y_{m-1}(t) - x_{m-1}(t) - 3 \sum_{i=0}^{m-1} x_i(t)x_{m-1-i}(t) - t(1 - \chi_m), \end{aligned} \tag{43}$$

In view of the m th-order deformation Eq. (37), we can obtain the following series solution

$$\begin{aligned} x(t) &= -\frac{2h_1(h_1^2 + 3h_1 + 3)}{\Gamma(2 + \alpha_1)} t^{\alpha_1+1} + \frac{2h_1h_2(h_1 + h_2 + 3)}{\Gamma(2 + \alpha_1 + \alpha_2)} t^{\alpha_1+\alpha_2+1} - \frac{4h_1^3\Gamma(3 + 2\alpha_1)}{\Gamma(2 + \alpha_1)^2\Gamma(3 + 3\alpha_1)} t^{3\alpha_1+2} + \dots, \\ y(t) &= \frac{2h_2(h_2^2 + 3h_2 + 3)}{\Gamma(2 + \alpha_2)} t^{\alpha_2+1} - \frac{4h_1h_2h_3}{\Gamma(2 + \alpha_1 + \alpha_2)} t^{\alpha_1+\alpha_2+1} - \frac{4h_2^2h_3}{\Gamma(3 + 2\alpha_2)} t^{2\alpha_2+1} + \dots, \\ z(t) &= 1 + \frac{2h_1h_3(h_1 + 3)}{\Gamma(2 + \alpha_1)} t^{\alpha_1+1} + \frac{2h_2h_3^2}{\Gamma(2 + \alpha_2)} t^{\alpha_2+2} - \frac{12h_1^2h_3}{\Gamma(2 + \alpha_1)^2} t^{2\alpha_1+2} - \frac{2h_1h_2h_3}{\Gamma(2 + \alpha_1 + \alpha_2)} t^{\alpha_1+\alpha_2+1} + \dots \end{aligned} \tag{44}$$

It is clear that the numerical results obtained using HAM is in good agreement with the exact solutions. This is illustrated by Fig. 6. Fig. 7 shows the HAM approximate solutions for various values of α_1 and α_2 .

5. Conclusions

The purpose of this paper is to construct the homotopy analysis method to systems of algebraic–differential equations of fractional order. A comparison with the results of other numerical method such as Adomian decomposition method indicates that the HAM is accurate, fast and reliable for such problems. There are some important points to make here. First, we have great freedom to choose the auxiliary parameters h_i and the auxiliary linear operators \mathcal{L}_i . Second, the homotopy analysis method was shown to be a simple, yet powerful analytic numeric scheme for handling systems of fractional order. Finally, generally speaking; the proposed approach can be further implemented to solve other nonlinear problems in the fractional calculus field.

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