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Analytic Study on Time-Fractional Schrödinger Equations: Exact Solutions by GDTM

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Abstract. In this work, we present a framework to obtain exact solutions to linear and nonlinear time-fractional Schrödinger equations. The generalized differential transform method (GDTM) is employed to derive analytical solutions for these equations. Some examples are tested and the results reveal that the technique introduced here is very effective and convenient for solving linear partial differential equations of fractional order.

1. Introduction

Mathematical modelling of many physical systems leads to linear and nonlinear fractional differential equations in various fields of physics and engineering. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Several mathematical methods including Adomian decomposition method [1-2], variational iteration method [3], homotopy perturbation method [4] and differential transform method [5] have been developed to obtain exact and approximate analytic solutions to differential equations of fractional order. These methods provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations.

The differential transform method was first introduced by Zhou [6] who solved linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in form of a polynomial. It is different from the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solution of ordinary or partial differential equations. The method is well addressed in [6-9]. Recently, the first two authors have developed a semi-numerical method, generalized differential transform method (GDTM), for solving linear partial differential equations of fractional order [5]. This method is based on the two-dimensional differential transform method [7, 8] and generalized Taylor's formula [9].

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The nonlinear Schrödinger equation is one of the most universal models that describe many physical nonlinear systems. Many methods are used to handle the equation such as Adomian decomposition method [11], variational iteration method [12,13], homotopy perturbation method [14], and other methods as well. In this work, the time-fractional Schrödinger equation

$$D_t^\alpha u + i u_{xx} + \gamma |u|^2 u = 0, \qquad u(x,0) = f(x), \qquad t > 0, \tag{1.1}$$

will be investigated by using GDTM, where γ is a constant, $u(x,t)$ is a complex function and $D^\alpha = D_0^\alpha$ is the Caputo fractional derivative of order α, where $0 < \alpha \le 1$. The Caputo fractional derivative [15,16] is defined as:

$$D_a^\alpha f(x) = J_a^{m-\alpha} D^m f(x), \tag{1.2}$$

where $m-1 < \alpha \le m$. Here D^m is the usual integer differential operator of order m and J_a^μ is the Riemann-Liouville integral operator of order $\mu > 0$, defined by

$$J_a^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)\,dt, \qquad x > a. \tag{1.3}$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem [15]. For more information on fractional derivatives and integrals one can consult the mentioned references.

2. Generalized two-dimensional differential transform method

Consider a function of two variables $u(x,y)$, and suppose that it can be represented as a product of two single-valued functions, i.e. $u(x,y) = f(x)\,g(y)$. Based on the properties of generalized two-dimensional differential transform [7,8], the function $u(x,y)$ can be represented as

$$\begin{aligned} u(x,y) &= \sum_{k=0}^{\infty} F_\alpha(k)(x-x_0)^{k\alpha} \sum_{h=0}^{\infty} G_\beta(k)(y-y_0)^{h\beta} \\ &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta}, \end{aligned} \tag{2.1}$$

where $0 < \alpha, \beta \le 1$, $U_{\alpha,\beta}(k,h) = F_\alpha(k) G_\beta(h)$ is called the spectrum of $u(x,y)$. The generalized two-dimensional differential transform of the function $u(x,y)$ is given as follows [5]

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)} \left[\left(D_{x_0}^\alpha\right)^k \left(D_{y_0}^\beta\right)^h u(x,y) \right]_{(x_0,y_0)}, \tag{2.2}$$

where $\left(D_{x_0}^\alpha\right)^k = D_{x_0}^\alpha D_{x_0}^\alpha \cdots D_{x_0}^\alpha$, k-times. In this paper, the lower case $u(x,y)$ represents the original function while the upper case $U_{\alpha,\beta}(k,h)$ stands for the transformed function. Based on the definitions (2.1) and (2.2), we have the following results:

Theorem 2.1 Suppose that $U_{\alpha,\beta}(k,h)$, $V_{\alpha,\beta}(k,h)$ and $W_{\alpha,\beta}(k,h)$ are the differential transformations of the functions $u(x,y)$, $v(x,y)$ and $w(x,y)$, respectively,

(a) if $u(x,y) = v(x,y) \pm w(x,y)$, then $U_{\alpha,\beta}(k,h) = V_{\alpha,\beta}(k,h) \pm W_{\alpha,\beta}(k,h)$,

(b) if $u(x, y) = a\,v(x, y)$, $a \in \boldsymbol{R}$, then $U_{\alpha,\beta}(k,h) = aV_{\alpha,\beta}(k,h)$,

(c) if $u(x, y) = v(x, y)\,w(x, y)$, then $U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(k-r,s)$,

(d) if $u(x, y) = (x-x_0)^{n\alpha}(y-y_0)^{m\beta}$, then $U_{\alpha,\beta}(k,h) = \delta(k-n)\delta(h-m)$,

(e) if $u(x, y) = D_{x_0}^{\alpha} v(x, y)$, $0<\alpha\le 1$, then $U_{\alpha,\beta}(k,h) = \dfrac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+1,h)$.

Theorem 2.2 If $u(x, y) = f(x)\,g(y)$ and the function $f(x) = x^{\lambda}\,h(x)$, where $\lambda > -1$, $h(x)$ has the generalized Taylor series expansion $h(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{\alpha n}$, and

(a) $\beta < \lambda + 1$ and α arbitrary

or

(b) $\beta \ge \lambda + 1$, α arbitrary and $a_n = 0$ for $n = 0,1,\cdots,m-1$, where $m-1<\beta\le m$,

then the generalized differential transform (2.2) becomes

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[D_{x_0}^{\alpha k}\left(D_{y_0}^{\beta}\right)^{h} u(x,y)\right]_{(x_0,y_0)}. \tag{2.3}$$

Theorem 2.3 If $v(x, y) = f(x)\,g(y)$, the function $f(x)$ satisfies the conditions given in Theorem 2.2, and $u(x,y) = D_{x_0}^{\gamma}\,v(x,y)$, then

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+1)+\gamma)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+\gamma/\alpha, h). \tag{2.4}$$

The proof of Theorems 2.1, 2.2 and 2.3 are given in [5]. An important observation that can be made here is that the GDTM assumes that the solution for a partial differential equation of fractional order can be written as a product of single-valued functions.

3. Solution of the problem

Setting $u(x,t) = v(x,t) + i\,w(x,t)$ and $f(x) = g_1(x) + i\,g_2(x)$ in Eqs. (1.1) and (1.2) leads to the following coupled system of equations

$$\begin{aligned} D_t^{\alpha}\, v + w_{xx} + \gamma(v^2+w^2)\,w &= 0, \\ D_t^{\alpha}\, w - v_{xx} - \gamma(v^2+w^2)\,v &= 0, \end{aligned} \tag{3.1}$$

subject to the initial conditions

$$v(x,0) = g_1(x)\ , \qquad w(x,0) = g_2(x), \tag{3.2}$$

where $0<\alpha\le 1$ and D^{α} is the Caputo fractional derivative of order α. To apply the method, we assume that the solutions $v(x,t)$ and $w(x,t)$ can be represented as a product of single-valued functions. Applying the generalized two-dimensional differential transform to both sides of the system (3.1), we obtain the following system of recurrence relations

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}V(k,h+1)+(k+1)(k+2)W(k+2,h)+\gamma\sum_{r=0}^{k}\sum_{s=0}^{h}$$

$$\left[\sum_{p=0}^{r}\sum_{q=0}^{h-s}\left(V(p,h-s-q)V(r-p,q)+W(p,h-s-q)W(r-p,q)\right)\right]W(k-r,s)=0,$$

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}W(k,h+1)-(k+1)(k+2)V(k+2,h)-\gamma\sum_{r=0}^{k}\sum_{s=0}^{h} \tag{3.3}$$

$$\left[\sum_{p=0}^{r}\sum_{q=0}^{h-s}\left(V(p,h-s-q)V(r-p,q)+W(p,h-s-q)W(r-p,q)\right)\right]V(k-r,s)=0.$$

Computing the generalized two-dimensional differential transforms of the initial conditions (3.2) and utilizing the recurrence relations (3.3), we obtain the solution.

4. Applications and results

In this section we consider a few examples that demonstrate the performance and efficiency of the generalized differential transform method for solving the time-fractional Schrödinger equations

Example 4.1 Consider the linear time-fractional Schrödinger equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+iu_{xx}=0,\quad u(x,0)=\cosh(2x),\quad t>0, \tag{4.1}$$

where $0<\alpha\le 1$. The generalized differential transforms of the initial conditions $v(x,0)=\cosh(2x)$ and $w(x,0)=0$ are given by

$$V(k,0)=\begin{cases}\frac{(2)^k}{k!}, & k\ even\\ 0, & k\ odd\end{cases},\qquad W(k,0)=0. \tag{4.2}$$

Consequently, solving the recurrence relations given in the system (3.3), where $\gamma=0$, by using (4.2), for $k\ge 0$ and $h>0$ we obtain

$$V(k,h)=\begin{cases}\frac{(2)^k}{k!}\cdot\frac{(-1)^{h/2}4^h}{\Gamma(\alpha h+1)}, & k,h\ even\\ 0, & otherwise\end{cases}, \tag{4.3}$$

$$W(k,h)=\begin{cases}\frac{(2)^k}{k!}\cdot\frac{(-1)^{(h-1)/2}4^h}{\Gamma(\alpha h+1)}, & k\ even,\ h\ odd\\ 0, & otherwise\end{cases}. \tag{4.4}$$

Therefore, using (3.4), the exact solution of Eq. (4.1)

$$u(x,t)=\cosh(2x)\left(\cos(4t^\alpha,\alpha)+i\sin(4t^\alpha,\alpha)\right), \tag{4.5}$$

is readily obtained, where the functions $\cos(z,\alpha)$ and $\sin(z,\alpha)$ are defined as

$$\cos(z,\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{\Gamma(2n\alpha+1)}, \qquad \sin(z,\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{\Gamma((2n+1)\alpha+1)}.$$

Example 4.2 Consider the nonlinear time-fractional Schrödinger equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+iu_{xx}+2|u|^2u=0, \qquad u(x,0)=\exp(ix), \qquad t>0, \tag{4.6}$$

where $0<\alpha\le 1$. The generalized differential transforms of the initial conditions $v(x,0)=\cos(x)$ and $w(x,0)=\sin(x)$ are given by

$$V(k,0)=\begin{cases}\frac{(-1)^{k/2}}{k!}, & k\ even\\ 0, & k\ odd\end{cases}, \qquad W(k,0)=\begin{cases}\frac{(-1)^{(k-1)/2}}{k!}, & k\ odd\\ 0, & k\ even\end{cases}. \tag{4.7}$$

Consequently, solving the recurrence relations given in the system (3.3), where $\gamma=2$, by using (4.7), for $k\ge 0$ and $h>0$ we obtain

$$V(k,h)=\begin{cases}\frac{(-1)^{k/2}}{k!}\cdot\frac{(-1)^{h/2}}{\Gamma(\alpha h+1)}, & k,h\ even\\ \frac{(-1)^{k/2}}{k!}\cdot\frac{(-1)^{h/2}}{\Gamma(\alpha h+1)}, & k,h\ odd\\ 0, & otherwise\end{cases}, \tag{4.8}$$

$$W(k,h)=\begin{cases}\frac{(-1)^{k/2}}{k!}\cdot\frac{(-1)^{(h-1)/2}}{\Gamma(\alpha h+1)}, & k\ even,\ h\ odd\\ \frac{(-1)^{(k-1)/2}}{k!}\cdot\frac{(-1)^{h/2}}{\Gamma(\alpha h+1)}, & k\ odd,\ h\ even\\ 0, & otherwise\end{cases}. \tag{4.9}$$

Therefore, using (3.4), the exact solution of Eq. (4.6)

$$u(x,t)=\cos(x)\cos(t^\alpha,\alpha)-\sin(x)\sin(t^\alpha,\alpha)+i\big(\sin(x)\cos(t^\alpha,\alpha)+\cos(x)\sin(t^\alpha,\alpha)\big), \tag{4.10}$$

is readily obtained. In case of $\alpha=1$, the solution of Eq. (4.6) reduces to $u(x,t)=\exp(i\,(x+t))$ which is exactly the same solution obtained in [12] using variational iteration method.

Example 4.3 Consider the nonlinear time-fractional Schrödinger equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+iu_{xx}+\frac{1}{2}|u|^2u=0, \qquad u(x,0)=\exp(ix), \qquad t>0, \tag{4.11}$$

where $0<\alpha\le 1$. Following the same manner as discussed in the previous examples, we obtain the following solution

$$u(x,t)=\cos(x)\cos(t^\alpha/2,\alpha)-\sin(x)\sin(t^\alpha/2,\alpha)+i\big(\sin(x)\cos(t^\alpha/2,\alpha)+\cos(x)\sin(t^\alpha/2,\alpha)\big). \tag{4.12}$$

In case of $\alpha = 1$, the solution of Eq. (4.11) reduces to $u(x,t) = \exp(i\,(x + t/2))$ which is exactly the same solution obtained in [14] using homotopy perturbation method.

5. Conclusions

The main goal of this work has been to derive analytical solutions for time-fractional Schrödinger equations. We have achieved this goal by using the GDTM. The method gives several successive approximations through using the recurrence relations of the coupled system of Schrödinger equation. A clear conclusion can be drawn from the results that the GDTM provides an efficient method to handle nonlinear partial differential equations of fractional order.

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