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An effective variational iteration algorithm for solving Riccati differential equations

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ABSTRACT

The piecewise variational iteration method (VIM) for solving Riccati differential equations (RDEs) provides a solution as a sequence of iterates. Therefore, its application to RDEs leads to the calculation of terms that are not needed and more time is consumed in repeated calculations for series solutions. In order to overcome these shortcomings, we propose an easy-to-use piecewise-truncated VIM algorithm for solving the RDEs. Some examples are given to demonstrate the simplicity and efficiency of the proposed method. Comparisons with the classical fourth-order Runge–Kutta method (RK4) verify that the new method is very effective and convenient for solving Riccati differential equations.

1. Introduction

The Riccati differential equations (RDEs) of the following form:

$$\begin{cases} u'(t) = A(t) + B(t)u(t) + C(t)u^2(t), & t_0 \le t \le T, \\ u(t_0) = c, \end{cases} \tag{1}$$

where $A(t)$, $B(t)$ and $C(t)$ are given functions and c is an arbitrary constant, are a class of nonlinear differential equations of much importance, and play a significant role in many fields of applied science [1]. For instance, solitary wave solutions of a nonlinear partial differential equation can be expressed as a polynomial in two elementary functions satisfying a projective Riccati equation [2]. Such problems also arise in the optimal control literature. However, deriving an analytical solution in an explicit form seems to be unlikely to be achievable except for certain special situations [3]. Of course, if one knows a particular solution, then the general solution can be easily derived. For general cases, one must appeal to numerical techniques or approximate approaches for getting the solutions. Therefore, the problem has attracted much attention and has been studied by many authors (see e.g. [4–7] and the references cited therein).

The variational iteration method, which was proposed originally by He [8–10], has been proved by many authors to be a powerful mathematical tool for various kinds of linear and nonlinear problems [8–19]. Though the VIM leads to fast converging solutions, there is unnecessary calculation in the solution procedure. He and Wu have suggested an effective technique in [9]. They truncate the approximate solution in each of the iterations of the VIM. Also, in order to accelerate the rate of convergence, various other modifications were suggested, for example, variational iteration—the Padé method [20], variational iteration—the Adomian method [11] and variational iteration—the differential transform method [21]. In this

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direction, Abassy, El-Tawil and Zoheiry suggested another effective modification [22]. Most interestingly, He and Wu in [9] have carefully studied trends and developments in the use of the VIM for solving nonlinear problems. Moreover, He et al. in [23] proposed new algorithms based on the VIM, which can overcome the shortcomings arising in the classical VIM. Also, Hersanu and Marinca suggested an optimal variational iteration algorithm for nonlinear oscillators [24].

Most recently, Geng et al. in [13] proposed a piecewise VIM in solving Eq. (1) in order to enlarge the convergence region and enhance the rate of the series solution. Since the methods presented in [13] and [23] (i.e., piecewise VIM and the VIM algorithms I, II and III, respectively) provide the solution as a sequence of iterates, their successive iterations may be very complex, so the resulting integrations in their iterative relations may be impossible to perform analytically. Moreover, their application to Eq. (1) leads to the calculation of terms that are not needed and repeated terms, and more time is consumed in unnecessary and repeated calculations for series solutions. In this work, we will overcome the disadvantages of the original and piecewise VIM for solving Eq. (1) by introducing a piecewise-truncated VIM. The method presented stops the calculations of terms that are not needed and repeated terms. Furthermore, it is effective in saving time and calculations.

2. The basic idea of the VIM

In this section, we describe the VIM for solving Eq. (1). The VIM method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise approximations can be used for numerical purposes. According to the VIM [8,9], we can construct a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^{t} \lambda_{(t,s)} \left\{ u_n'(s) - A(s) - B(s)\tilde{u}_n(s) - C(s)\tilde{u}_n^2(s) \right\} \mathrm{d}s, \tag{2}$$

where $u_0(t) = u(t_0) = c$ is the initial guess, $\lambda_{(t,s)} \neq 0$ is a general multiplier, which can be identified optimally via the variational theory [10], the subscript n denotes the nth approximation, and $\tilde{u}_n$ and $\tilde{u}_n^2$ are considered as restricted variations, i.e., $\delta\tilde{u}_n^2, \tilde{u}_n = 0$. Making the above correction functional stationary, we can readily obtain the stationary conditions and then the Lagrange multiplier, which is $\lambda_{(t,s)} = -1$. Therefore, we have the following variational iteration formula:

$$u_{n+1}(t) = u_n(t) - \int_{t_0}^{t} \left\{ u_n(s) - A(s) - B(s)u_n(s) - C(s)u_n^2(s) \right\} \mathrm{d}s. \tag{3}$$

Accordingly, the successive approximations $u_n(t)$, $n \geq 0$, of the solution $u(t)$ will be readily obtained by selecting the zeroth component. Consequently, the exact solution may be obtained by using $u(t) = \lim_{n\to\infty} u_n(t)$. We point out that the convergence of the VIM was discussed in [25].

3. The basic idea of the piecewise-truncated VIM

In general, the application of the VIM to the RDEs leads to the calculation of terms that are not needed and repeated terms. In order to completely eliminate these unnecessary and repeated calculations, a modification of the VIM must be made [22,23]. Here, we assume that the right hand side of (1) in each of iterations of the VIM is expanded in a Taylor series around t_0. By simple integration by parts, the variational iteration formula (3) can be written as below [9,23]:

$$u_{n+1}(t) = u_0(t) + \int_{t_0}^{t} F_n(s)\mathrm{d}s, \quad n \geq 0, \tag{4}$$

where $u_0(t) = c$ and $F_n(s)$ is calculated from the relation

$$A(s) + B(s)u_n(s) + C(s)u_n^2(s) = F_n(s) + \mathrm{O}[(s - t_0)^{n+1}]. \tag{5}$$

Let us rewrite (4) as the following iteration formula:

$$u_{n+1}(t) = u_0(t) + \int_{t_0}^{t} F_{n-1}(s)\mathrm{d}s + \int_{t_0}^{t} [F_n(s) - F_{n-1}(s)]\mathrm{d}s. \tag{6}$$

But it is known from (4) that

$$u_n(t) = u_0(t) + \int_{t_0}^{t} F_{n-1}(s)\mathrm{d}s. \tag{7}$$

Substituting (7) in (6), we get the following improved formula [23], which is called the truncated VIM (TV):

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^{t} \{F_n(s) - F_{n-1}(s)\}\,\mathrm{d}s, \quad n \geq 0, \tag{8}$$

where $F_{-1}(s) = 0$, and $F_n(s)$, $n \geq 0$, are obtained from (5).

Recently, He et al. [23] proposed three standard variational iteration algorithms for solving differential equations of different types. Eq. (4) is generally called the variational iteration algorithm II, and Eq. (8) the variational iteration algorithm III. For more details on the derivation and applications of these algorithms see [23].

It is worth pointing out that using the TV formula (8) one can avoid all the repetition of calculations, and calculations of terms that are not needed. Also, it can be used for solving nonlinear RDEs with complicated variable coefficients. Furthermore, it can reduce the size of calculations. Besides this, the TV algorithm (8) solves a nonlinear RDE exactly if its solution is an algebraic polynomial up to some degree.

By using the TV algorithm (8), we obtain a series solution, which in practice is a truncated series solution. Unfortunately, this series solution gives a good approximation to the exact solution only in a small region of t. An easy and reliable way of ensuring the validity of the approximations (8) for large t is to determine the solution in a sequence of equal subintervals t, i.e. $I_i = [t_i, t_{i+1}]$ where $\Delta t = t_{i+1} - t_i$, $i = 0, 1, 2, \ldots, N-1$, with $t_N = T$. According to [13,23], therefore, we can obtain the following n_{i+1}th-order approximation $u_{i+1,n_{i+1}}(t)$ on I_i for Eq. (8), which is called the piecewise TV (PTV):

$$\begin{cases} u_{i+1,k+1}(t) = u_{i+1,k}(t) + \displaystyle\int_{t_i}^{t} \left\{F_{i+1,k}(s) - F_{i+1,k-1}(s)\right\} \mathrm{d}s, \quad k = 0, 1, \ldots, n_{i+1}-1,\ t \in [t_i, t_{i+1}], \\ u_{i+1,0}(t) = u_{i,n_i}(t_i) = c_i, \\ A(s) + B(s)u_{i+1,k}(s) + C(s)u_{i+1,k}^2(s) = F_{i+1,k}(s) + \mathrm{O}\left((s-t_i)^{k+1}\right), \quad i = 0, 1, \ldots, N-1, \end{cases} \tag{9}$$

where $u_{0,n_0}(t_0) = u(t_0) = c_0$ and $F_{i+1,-1}(s) = 0$, $0 \le i \le N-1$. Thus, in the light of (9), the approximation of Eq. (1) on the entire interval $[t_0, T]$ can easily be obtained. Following the present section, the n_{i+1}th-order approximate analytical solution via the PTV method for Eq. (1) can be written as

$$u_{i+1,n_{i+1}}(t) = \sum_{m=0}^{n_{i+1}} \frac{\gamma_{i,m}(t_i, c_i)}{m!}(t-t_i)^m + \mathrm{O}[(t-t_i)^{n_{i+1}+1}], \quad t \in I_i = [t_i, t_{i+1}], \tag{10}$$

where $\gamma_{i,m}(t_i, c_i)$ is a coefficient dependent on t_i and c_i. The expression (10) demonstrates that the n_{i+1}th-order PTV method has an error per step of the order of $(\Delta t)^{n_{i+1}+1}$, while the total accumulated error is of order $(\Delta t)^{n_{i+1}}$.

4. Illustrative examples

To give a clear overview of the content of this study, several RDEs will be tested with the above-mentioned algorithm, which will ultimately show the efficiency and accuracy of this method. All the results here are computed using Maple 11.

Example 4.1. As the first example, we consider a RDE with constant coefficients as follows [4–7,11–13]:

$$u'(t) = 1 + 2u(t) - u^2(t), \quad 0 \le t \le 100, \tag{11}$$

with the initial condition $u(0) = 0$. The exact solution of (11) was found to be

$$u(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \tag{12}$$

In order to solve this equation by using the PTV algorithm, according to (9), we can obtain a fourth-order PTV approximation in the subintervals I_i as follows:

$$\begin{aligned} u_{i+1,4}(t) &= c_i + (1+2c_i-c_i^2)(t-t_i) + (1-c_i)(1+2c_i-c_i^2)(t-t_i)^2 \\ &\quad + \frac{1}{3}(3c_i^2+1-6c_i)(1+2c_i-c_i^2)(t-t_i)^3 + \frac{1}{3}(1-c_i)(3c_i^2-1-6c_i)(1+2c_i-c_i^2)(t-t_i)^4, \end{aligned} \tag{13}$$

where $t_0 = 0$, $c_0 = u(0) = 0$ and $c_{i+1} = u_{i+1,4}(t_{i+1})$, $i = 0, 1, \ldots, N-1$. The numerical results for (13) when $\Delta t = 0.1$ ($N = 1000$) are shown in Fig. 1. Fig. 1 exhibits a comparison of the approximation obtained using the fourth-order PTV method with the exact solutions in a sample interval of t, i.e. $0 \le t \le 100$. It is easy to deduce, from the numerical results in Fig. 1, that our approximate solution using the PTV algorithm is in excellent agreement with the exact value in the large interval of t.

Example 4.2. Now we consider a RDE with uncomplicated variable coefficients as below [5,13]:

$$u'(t) = 1 + t^2 - u^2(t), \quad 0 \le t \le 100, \tag{14}$$

with the initial condition $u(0) = 1$ and the exact solution

$$u(t) = t + \frac{2\mathrm{e}^{-t^2}}{2 + \sqrt{\pi}\,\mathrm{erf}(t)} \tag{15}$$

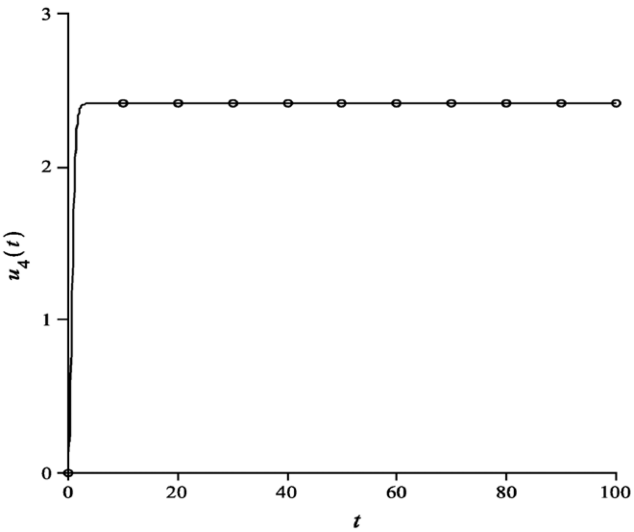

Fig. 1. Approximate solution for Example 4.1 using the fourth-order PTV algorithm when $\Delta t = 0.1$, where the fourth-order PTV solution is given by the circle symbols and the exact solution is given by the solid line.

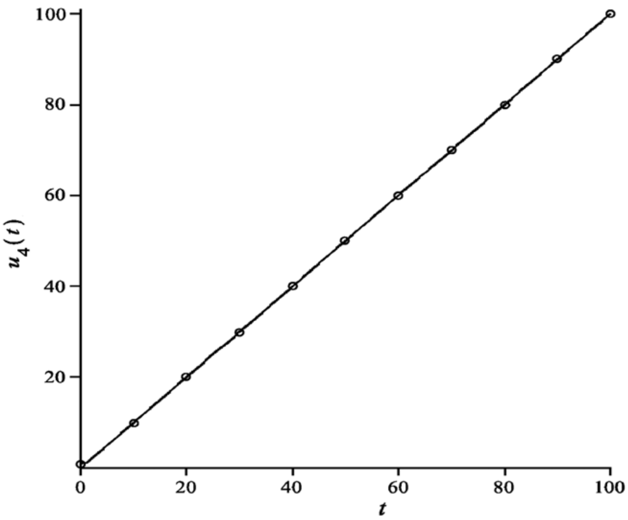

Fig. 2. Approximate solution for Example 4.2 using the fourth-order PTV algorithm when $\Delta t = 0.1$, where the fourth-order PTV solution is given by the circle symbols and the exact solution is given by the solid line.

where $\mathrm{erf}(t)$ is the error function. According to (9), taking $n_{i+1} = 4,\ i = 0, 1, \ldots, N-1$, and $N = 1000$ ($\Delta t = 0.1$), we can obtain the approximations of (14) on $[0, 100]$. The numerical results are shown in Fig. 2. It is easy to conclude that the PTV algorithm is much more efficient and accurate.

Example 4.3. As the final example, we consider a RDE with the complicated variable coefficients as follows:

$$u'(t) = \frac{t\mathrm{e}^{-t^2}}{1+t^2} + \frac{\tanh(t)}{\sqrt{1+t}}u(t) - \ln(1+t)\mathrm{e}^{\sin(t)}u^2(t), \quad 0 \le t \le 100, \tag{16}$$

with the initial condition $u(0) = 1$. According to (9), taking $n_{i+1} = 4,\ i = 0, 1, \ldots, N-1$, and $N = 1000$ ($\Delta t = 0.1$), we can obtain the approximations of (16) on $[0, 100]$. The numerical results are shown in Fig. 3.

As shown in Fig. 3, our approximate solution using the PTV algorithm is in superior agreement with the numerical values in the large interval.

Remark 4.1. Two important points can be made here. First, both the PTV algorithm and the numerical RK4 method, with the same step size 0.1 in solving the given illustrative examples, have stable numerical outcomes (e.g., see Fig. 4). Second,

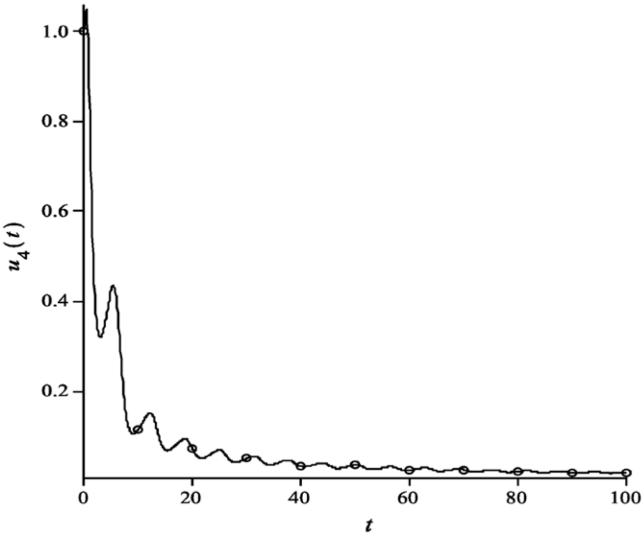

Fig. 3. Approximate solution for Example 4.3 using the fourth-order PTV algorithm when $\Delta t = 0.1$, where the fourth-order PTV solution is given by the solid line and the numerical RK4 ($h = 0.01$) solution is given by the circle symbols.

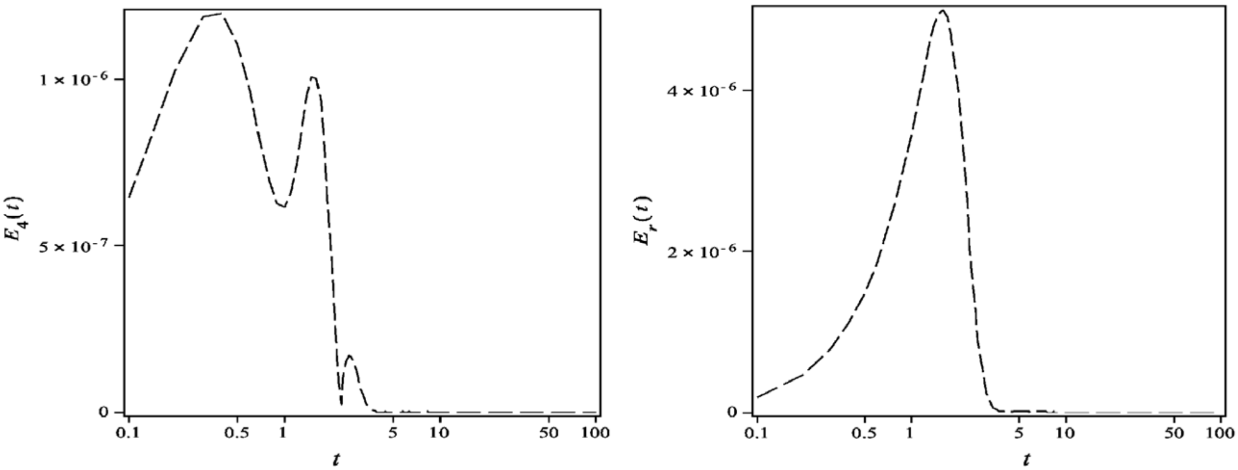

Fig. 4. The absolute errors of the fourth-order PTV solution (left) when $\Delta t = 0.1$ ($E_4(t) = |u_{RK78}(t) - u_4(t)|$) and the numerical RK4 solution (right) when $h = 0.1$ ($E_r(t) = |u_{RK78}(t) - u_{RK4}(t)|$) for Example 4.2.

the PTV algorithm when $n_{i+1} = 4,\ i = 0, 1, \dots, N-1$, gives an approximate analytical solution which is of comparable accuracy to that obtained by the numerical RK4 method (e.g., observe Fig. 4).

Also, it seems, from the above comparative results and in the light of (10), that the PTV approximations for $n_{i+1} > 4,\ i = 0, 1, \dots, N-1$, converge faster than the numerical RK4 method. It is easy to conclude that the PTV algorithm could lead to a promising analytical tool for solving nonlinear ordinary differential equations.

5. Concluding notes

In this work, we proposed an effective and easy-to-use analytical algorithm based on the VIM technique called the piecewise-truncated VIM (PTV) for accurately solving the RDEs. We tested several modeling equations by using the PTV algorithm presented in this paper, and the results obtained have shown excellent performance. Comparison with the numerical RK methods confirms the very high accuracy of the algorithm presented. The numerical results reveal that the PTV algorithm is easy to implement, accurate when applied to the RDEs and avoids tedious computational work. Furthermore, it can further be employed to solve a large class of ordinary differential equations with high accuracy.

References

[1] W.T. Reid, Riccati Differential Equations, Academic Press, New York, 1972.
[2] J.F. Carinena, G. Marmo, A.M. Perelomov, M.F. Ranada, Related operators and exact solutions of Schrödinger equations, Internat. J. Modern Phys. A 13 (1998) 4913–4929.
[3] M.R. Scott, Invariant Imbedding and its Applications to Ordinary Differential Equations, Addison-Wesley, 1973.
[4] Y. Tan, S. Abbasbandy, Homotopy analysis method for quadratic Riccati differential equation, Commun. Nonlinear Sci. Numer. Simul. 13 (2008) 539–546.
[5] B.Q. Tang, X.F. Li, A new method for determining the solution of Riccati differential equations, Appl. Math. Comput. 194 (2007) 431–440.
[6] S. Abbasbandy, Iterated He's homotopy perturbation method for quadratic Riccati differential equation, Appl. Math. Comput. 175 (2006) 581–589.
[7] M.A. El-Tawil, A.A. Bahnasawi, A. Abdel-Naby, Solving Riccati differential equation using Adomian's decomposition method, Appl. Math. Comput. 157 (2004) 503–514.
[8] J.H. He, Variational iteration method—a kind of non-linear analytical technique: some examples, Internat. J. Non-Linear Mech. 34 (1999) 699–708.
[9] J.H. He, X.-H. Wu, Variational iteration method: new development and applications, Comput. Math. Appl. 54 (2007) 881–894.
[10] J.H. He, Variational principle for some nonlinear partial differential equations with variable coefficients, Chaos Solitons Fractals 19 (2004) 847–851.
[11] S. Abbasbandy, A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials, J. Comput. Appl. Math. 207 (2007) 59–63.
[12] B. Batiha, M.S.M. Noorani, I. Hashim, Application of variational iteration method to a general Riccati equation, Int. Math. Forum 2 (2007) 2759–2770.
[13] F. Geng, Y. Lin, M. Cui, A piecewise variational iteration method for Riccati differential equations, Comput. Math. Appl. 58 (2009) 2518–2522.
[14] J.H. He, Some asymptotic methods for strongly nonlinear equations, Internat. J. Modern Phys. B 20 (2006) 1141–1199.
[15] J.H. He, An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering, Internat. J. Modern Phys. B 22 (2008) 3487–3578.
[16] S.M. Goh, M.S.N. Noorani, I. Hashim, M.M. Al-Sawalha, Variational iteration method as a reliable treatment for the hyperchaotic Rossler system, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 363–371.
[17] R. Mokhtari, M. Mohammadi, Some remarks on the variational iteration method, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 67–74.
[18] C. Chun, Variational iteration method for a reliable treatment of heat equations with ill-defined initial data, Int. J. Nonlinear Sci. Numer. Simul. 9 (2008) 435–440.
[19] M.A. Noor, S.T. Mohyud-Din, Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials, Int. J. Nonlinear Sci. Numer. Simul. 9 (2008) 141–156.
[20] T.A. Abassy, M.A. El-Tawil, H. El Zoheiry, Solving nonlinear partial differential equations using the modified variational iteration—Padé technique, J. Comput. Appl. Math. 207 (2007) 73–91.
[21] N. Bildik, A. Konuralp, The use of variational iteration method, differential transform method and adomian decomposition method for solving different types of nonlinear partial differential equations, Int. J. Nonlinear Sci. Numer. Simul. 7 (2006) 65–70.
[22] T.A. Abassy, M.A. El-Tawil, H. El Zoheiry, Toward a modified variational iteration method, J. Comput. Appl. Math. 207 (2007) 137–147.
[23] J.H. He, G.C. Wu, F. Austin, The variational iteration method which should be followed, Nonlinear Sci. Lett. A 1 (2010) 1–30.
[24] N. Herişanu, V. Marinca, A modified variational iteration method for strongly nonlinear problems, Nonlinear Sci. Lett. A 1 (2010) 183–192.
[25] M. Tatari, M. Dehghan, On the convergence of He's variational iteration method, J. Comput. Appl. Math. 207 (2007) 121–128.