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A reliable algorithm of homotopy analysis method for solving nonlinear fractional differential equations

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ABSTRACT

In this paper, based on the homotopy analysis method (HAM), a powerful algorithm is developed for the solution of nonlinear ordinary differential equations of fractional order. The proposed algorithm presents the procedure of constructing the set of base functions and gives the high-order deformation equation in a simple form. Different from all other analytic methods, it provides us with a simple way to adjust and control the convergence region of solution series by introducing an auxiliary parameter $\hbar$. The analysis is accompanied by numerical examples. The algorithm described in this paper is expected to be further employed to solve similar nonlinear problems in fractional calculus.

1. Introduction

In the past decades, both mathematicians and physicists have devoted considerable effort to find robust and stable numerical and analytical methods for solving fractional differential equations of physical interest. Numerical and analytical methods have included finite difference method [1–3], Adomian decomposition method [4–8], variational iteration method [9–12], homotopy perturbation method [13–16], generalized differential transform method [17–20], and homotopy analysis method [21,22]. Among them, the homotopy analysis method (HAM) [21–31] provides an effective procedure for explicit and numerical solutions of a wide and general class of differential systems representing real physical problems. Based on homotopy, which is a basic concept in topology, the validity of the HAM is independent of whether or not there exist small parameters in the considered equation. Besides, different from all previous numerical and analytical methods, it provides us with a simple way to adjust and control the convergence of solution series. Especially, it provides us with great freedom to replace a nonlinear differential equation of order n into an infinite number of linear differential equations of order k, where the order k is even unnecessary to be equal to the order n. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a possibility to analyze strongly nonlinear problems.

The motivation of this paper is to extend the application of the homotopy analysis method proposed by Liao [23–28] to solve nonlinear differential equations of fractional order. By means of introducing an auxiliary parameter $\hbar$, we can adjust and control the convergence region of solution series. We also show that the procedure of constructing the set of base functions and the high-order deformation equation in a simple form.

There are several definitions of a fractional derivative of order $\alpha > 0$ [32,33]. The two most commonly used definitions are Riemann–Liouville and Caputo. Each definition uses Riemann–Liouville fractional integration and derivatives of whole order.

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The difference between the two definitions is in the order of evaluation. Riemann–Liouville fractional integration of order α is defined as,

$$J^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \quad \alpha > 0,\ x > 0. \tag{1}$$

The next two equations define Riemann–Liouville and Caputo fractional derivatives of order α, respectively,

$$D^{\alpha}f(x) = \frac{d^m}{dx^m}(J^{m-\alpha}f(x)), \tag{2}$$

$$D_*^{\alpha}f(x) = J^{m-\alpha}\left(\frac{d^m}{dx^m}f(x)\right), \tag{3}$$

where $m-1 < \alpha \leqslant m$ and $m \in N$. For now, Caputo fractional derivative will be denoted by D_*^{α} to maintain a clear distinction with Riemann–Liouville fractional derivative.

Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. Riemann–Liouville fractional derivative is computed in the reverse order. Therefore, Caputo fractional derivative allows traditional initial and boundary conditions to be included in the formulation of the problem, but Riemann–Liouville fractional derivative allows initial conditions in terms of fractional integrals and their derivatives. For homogeneous initial condition assumption, these two operators coincide. Through out this paper the unknown function $u(t)$ is assumed to be a causal function of time, and the time-fractional derivative is considered in Caputo sense.

2. Basic idea of homotopy analysis method

In this section the basic ideas of the homotopy analysis method are introduced. Here a description of the method [26] is given to handle the general nonlinear problem,

$$\mathcal{N}(u(t)) = 0, \quad t > 0, \tag{4}$$

where $\mathcal{N}$ is a nonlinear operator and $u(t)$ is unknown function of the independent variable t.

2.1. Zero-order deformation equation

Let $u_0(t)$ denote an initial guess of the exact solution of Eq. (4), $\hbar \neq 0$ an auxiliary parameter, $H(t) \neq 0$ an auxiliary function, and $\mathcal{L}$ an auxiliary linear operator with the property,

$$\mathcal{L}(f(t)) = 0 \quad \text{when } f(t) = 0. \tag{5}$$

The auxiliary parameter $\hbar$, the auxiliary function $H(t)$, and the auxiliary linear operator $\mathcal{L}$ play important roles within the HAM to adjust and control the convergence region of solution series. Liao [26] constructs, using $q \in [0,1]$ as an embedding parameter, the so-called zero-order deformation equation,

$$(1-q)\mathcal{L}[\Phi(t;q) - u_0(t)] = q\hbar H(t)\mathcal{N}[\Phi(t;q)], \tag{6}$$

where $\Phi(t;q)$ is the solution which depends on $\hbar$, $H(t)$, $\mathcal{L}$, $u_0(t)$ and q. When $q = 0$, the zero-order deformation Eq. (6) becomes,

$$\Phi(t;0) = u_0(t), \tag{7}$$

and when $q = 1$, since $\hbar \neq 0$ and $H(t) \neq 0$, then the zero-order deformation Eq. (6) reduces to,

$$\mathcal{N}[\Phi(t;1)] = 0. \tag{8}$$

So, $\Phi(t;1)$ is exactly the solution of the nonlinear Eq. (4). Define the so-called mth order deformation derivatives,

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \Phi(t;q)}{\partial q^m}\bigg|_{q=0}. \tag{9}$$

If the power series (9) of $\Phi(t;q)$ converges at $q = 1$, then we get the following series solution:

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t), \tag{10}$$

where the terms $u_m(t)$ can be determined by the so-called high-order deformation equations described below.

2.2. High-order deformation equation

Define the vector,

$$\vec{u}_n = \{u_0(t), u_1(t), u_1(t), \ldots, u_n(t)\}. \tag{11}$$

Differentiating Eq. (6) m times with respect to embedding parameter q, then setting $q = 0$ and dividing them by $m!$, we have the so-called mth-order deformation equation,

$$\mathscr{L}[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t) R_m(\vec{u}_m, t), \tag{12}$$

where,

$$\chi_m = \begin{cases} 0, & m \leqslant 1 \\ 1, & \text{otherwise} \end{cases}, \tag{13}$$

and

$$R_m(\vec{u}_{m-1}, t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathscr{N}[\Phi(t;q)]}{\partial q^{m-1}} \bigg|_{q=0}. \tag{14}$$

For any given nonlinear operator $\mathscr{N}$, the term $R_m(\vec{u}_{m-1}, t)$ can be easily expressed by (14). Thus, we can gain $u_1(t), u_2(t), \ldots$ by means of solving the linear high-order deformation Eq. (12) one after the other in order. The mth-order approximation of $u(t)$ is given by,

$$u(t) = \sum_{k=0}^{m} u_k(t). \tag{15}$$

3. The reliable algorithm

The homotopy analysis method which provides an analytical approximate solution is applied to various nonlinear problems [21–31]. In this section, we present a reliable approach of the homotopy analysis method. This new modification can be implemented for integer order and fractional order nonlinear differential equations. To illustrate the basic ideas of the new algorithm, we consider the following nonlinear differential equation of fractional order:

$$D_*^\alpha u(t) = \mathscr{N}(u) + g(t), \quad t > 0, \tag{16}$$

where $m - 1 < \alpha \leqslant m$, $\mathscr{N}$ is a nonlinear operator which might include other fractional derivatives of order less than α, $g(t)$ is a known analytic function and D_*^α is the Caputo fractional derivative of order α.

In view of the homotopy technique, we can construct the following homotopy:

$$(1-q)\mathscr{L}[\phi(t,q) - \phi_0(t)] = q\hbar H(D_*^\alpha \phi(t,q) - \mathscr{N}[\phi(t,q)] - g(t)), \tag{17}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a non zero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, ϕ_0 is an initial guess of $u(t)$ and $\mathscr{L}$ is an auxiliary linear operator that may be defined as $\mathscr{L} = \frac{d^m}{dt^m}$ or $\mathscr{L} = \frac{d^\alpha}{dt^\alpha}$. When $q = 0$, Eq. (17) becomes,

$$\mathscr{L}[\phi(t,0) - \phi_0(t)] = 0. \tag{18}$$

Its obvious that when $q = 1$, Eq. (17) becomes the original nonlinear Eq. (16). Thus as q varies from 0 to 1, the solution $u(x,q)$ varies from the initial guess $u_0(t)$ to the solution $\phi(t,1)$. The basic assumption of the new approach is that the solution of Eq. (17) can be expressed as a power series in q,

$$\phi = \phi_0 + q\phi_1 + q^2\phi_2 + \cdots. \tag{19}$$

Substituting the series (19) into the homotopy (17) and then equating the coefficients of the like powers of q, we obtain the high-order deformation equations,

$$\begin{aligned} \mathscr{L}[\phi_1] &= \hbar H\big(D_*^\alpha \phi_0 - \mathscr{N}_0(\phi_0) - g(t)\big), \\ \mathscr{L}[\phi_2] &= \mathscr{L}[\phi_1] + \hbar H\big(D_*^\alpha \phi_1 - \mathscr{N}_1(\phi_0, \phi_1)\big), \\ \mathscr{L}[\phi_3] &= \mathscr{L}[\phi_2] + \hbar H\big(D_*^\alpha \phi_2 - \mathscr{N}_2(\phi_0, \phi_1, \phi_2)\big), \\ \mathscr{L}[\phi_4] &= \mathscr{L}[\phi_3] + \hbar H\big(D_*^\alpha \phi_3 - \mathscr{N}_3(\phi_0, \phi_1, \phi_2, \phi_3)\big), \\ &\vdots \end{aligned} \tag{20}$$

where,

$$\mathscr{N}(\phi_0 + q\phi_1 + q^2\phi_2 + \cdots) = \mathscr{N}_0(\phi_0) + q\mathscr{N}_1(\phi_0, \phi_1) + q^2\mathscr{N}_2(\phi_0, \phi_1, \phi_2) + \cdots.$$

The approximate solution of Eq. (16), therefore, can be readily obtained,

$$u = \lim_{q \to 1} \phi = \phi_0 + \phi_1 + \phi_2 + \cdots. \tag{21}$$

The success of the technique is based on the proper selection of the initial guess ϕ_0. Applying the operator J^α to both sides of Eq. (16) gives,

$$u(t) = \sum_{k=0}^{m-1} u^{(k)}(0+)\frac{t^k}{k!} + J^\alpha \mathcal{N}(u) + J^\alpha g(t), \quad t > 0. \tag{22}$$

Neglecting the nonlinear term $J^\alpha \mathcal{N}(u)$ on the right hand side, we can use the remaining part as the initial guess of the solution. That is,

$$\phi_0(t) = \sum_{k=0}^{m-1} u^{(k)}(0+)\frac{t^k}{k!} + J^\alpha g(t). \tag{23}$$

The set of base function can be obtained, based on $\phi_0(t)$ given in Eq. (23), as,

$$\{t^{m\alpha \pm n}, \quad m, n \in N\}.$$

The main advantage of the new modification, as we will see in the next section, is that its applicable to a wide class of nonlinear differential equations of fractional order and the set of base functions will be easily constructed.

4. Applications

In order to assess both the accuracy and the convergence order of the homotopy analysis method presented in this paper for fractional differential equations, we have applied it to the following two nonlinear problems.

Example 1. Consider the fractional Riccati equation,

$$D_*^\alpha u + u^2 = 1, \quad t > 0, \tag{24}$$

where $0 < \alpha \leqslant 1$, subject to the initial condition

$$u(0) = 0. \tag{25}$$

The exact solution, when $\alpha = 1$, is

$$u(t) = \frac{e^{2t} - 1}{e^{2t} + 1}, \tag{26}$$

and we can observe that, as $t \to \infty$, $u(t) \to 1$. In view of the algorithm presented in the previous section, if we select the auxiliary linear operator as $\mathscr{L} = \frac{d^\alpha}{dt^\alpha}$, we can construct the homotopy,

$$(1-q)\mathscr{L}[\phi(t,q) - \phi_0(t)] = q\hbar H(D_*^\alpha \phi(t,q) + \phi^2(t,q) - 1). \tag{27}$$

According to (23), we have the initial guess,

$$\phi_0 = \frac{t^\alpha}{\Gamma(\alpha+1)}. \tag{28}$$

Taking $H(t) = 1$ and substituting (19) and the initial guess (28) into the homotopy (27) then equating the terms with identical powers of q, we obtain the following set of linear differential equations of fractional order:

$$\begin{aligned}
q^1 &: \quad D_*^\alpha \phi_1 = \hbar(D_*^\alpha \phi_0 + \phi_0^2 - 1), \\
q^2 &: \quad D_*^\alpha \phi_2 = D_*^\alpha \phi_1 + \hbar(D_*^\alpha \phi_1 + 2\phi_0\phi_1), \\
q^3 &: \quad D_*^\alpha \phi_3 = D_*^\alpha \phi_2 + \hbar(D_*^\alpha \phi_2 + \phi_1^2 + 2\phi_0\phi_2), \\
&\quad \vdots \\
q^n &: \quad D_*^\alpha \phi_n = D_*^\alpha \phi_{n-1} + \hbar\left(D_*^\alpha \phi_{n-1} + \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty} \lambda^i \phi_i\right)^2\right]_{\lambda=0}\right).
\end{aligned} \tag{29}$$

Consequently, solving the above linear equations, the first few components of the homotopy analysis solution for Eq. (24) are derived as follows:

$$\begin{aligned}
\phi_1 &= \hbar \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} t^{3\alpha}, \\
\phi_2 &= \hbar \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} t^{3\alpha} + \hbar^2 \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} t^{3\alpha} + 2\hbar^2 \frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} t^{5\alpha}, \\
\phi_3 &= \hbar \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} t^{3\alpha} + 2\hbar^2 \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} t^{3\alpha} + \hbar^3 \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} t^{3\alpha} \\
&\quad + 4\hbar^2 \frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} t^{5\alpha} + 4\hbar^3 \frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} t^{5\alpha} \\
&\quad + \hbar^3 \frac{\Gamma(2\alpha+1)^2\Gamma(6\alpha+1)}{\Gamma(\alpha+1)^4\Gamma(3\alpha+1)^2\Gamma(7\alpha+1)} t^{7\alpha} + 4\hbar^3 \frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)}{\Gamma(\alpha+1)^4\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)} t^{7\alpha},
\end{aligned} \tag{30}$$

which are the same approximate solutions obtained in [21]. Now, if we select the auxiliary linear operator as $\mathscr{L} = \frac{d}{dt}$ and use the homotopy,

$$(1-q)\mathscr{L}[\phi(t,q) - \phi_0(t)] = q\hbar H(D_*^{\alpha}\phi(t,q) + \phi^2(t,q) - 1), \tag{31}$$

then we obtain, when $H(t) = 1$, the following set of linear differential equations:

$$\begin{aligned}
q^1: \quad & \frac{d}{dt}\phi_1 = h(D_*^{\alpha}\phi_0 + \phi_0^2 - 1),\\
q^2: \quad & \frac{d}{dt}\phi_2 = \frac{d}{dt}\phi_1 + h(D_*^{\alpha}\phi_1 + 2\phi_0\phi_1),\\
q^3: \quad & \frac{d}{dt}\phi_3 = \frac{d}{dt}\phi_2 + h(D_*^{\alpha}\phi_2 + \phi_1^2 + 2\phi_0\phi_2),\\
& \vdots\\
q^n: \quad & \frac{d}{dt}\phi_n = \frac{d}{dt}\phi_{n-1} + h\left(D_*^{\alpha}\phi_{n-1} + \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)^2\right]_{\lambda=0}\right).
\end{aligned} \tag{32}$$

Solving the above linear equations, we obtain,

$$\begin{aligned}
\phi_1 &= \hbar\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(2\alpha+2)}t^{2\alpha+1},\\
\phi_2 &= \hbar\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(2\alpha+2)}t^{2\alpha+1} + \hbar^2\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(\alpha+3)}t^{\alpha+2} + 2\hbar^2\frac{\Gamma(2\alpha+1)\Gamma(3\alpha+2)}{\Gamma(\alpha+1)^3\Gamma(2\alpha+2)\Gamma(3\alpha+3)}t^{3\alpha+2},\\
\phi_3 &= \hbar\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(2\alpha+2)}t^{2\alpha+1} + 2\hbar^2\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(\alpha+3)}t^{\alpha+2} + \hbar^3\frac{\Gamma(2\alpha+1)}{6\Gamma(\alpha+1)^2}t^3\\
&\quad + 4\hbar^2\frac{\Gamma(2\alpha+1)\Gamma(3\alpha+2)}{\Gamma(\alpha+1)^3\Gamma(2\alpha+2)\Gamma(3\alpha+3)}t^{3\alpha+2} + 2\hbar^3\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^3\Gamma(2\alpha+4)}\left(\frac{\Gamma(3\alpha+2)}{\Gamma(2\alpha+2)} + \frac{\Gamma(2\alpha+3)}{\Gamma(\alpha+3)}\right)t^{2\alpha+3}\\
&\quad + \hbar^3\frac{\Gamma(2\alpha+1)^2\Gamma(4\alpha+3)}{\Gamma(\alpha+1)^4\Gamma(2\alpha+2)^2\Gamma(4\alpha+4)}t^{4\alpha+3} + 4\hbar^3\frac{\Gamma(2\alpha+1)\Gamma(3\alpha+2)\Gamma(4\alpha+3)}{\Gamma(\alpha+1)^4\Gamma(2\alpha+2)\Gamma(3\alpha+3)\Gamma(4\alpha+4)}t^{4\alpha+3}.
\end{aligned} \tag{33}$$

Setting $\hbar = -1$ in Eqs. (30) and (33), the above expressions are exactly the same as those given by the Adomian decomposition method and homotopy perturbation method. This illustrates that the two methods are indeed special cases of the homotopy analysis method. However, mostly, the results given by the Adomian decomposition method and homotopy perturbation method converge to the corresponding numerical solutions in a rather small region, as shown in Fig. 1. But, different from those two methods, the homotopy analysis method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter $\hbar$. Figs. 2 and 3 show the approximate solutions (30) and (33) for different values of $\hbar$ when $\alpha = 0.5$, respectively. Moreover, the homotopy-Padé technique can greatly accelerate the convergence of the HAM solutions as shown in Fig. 4. It is found that the $[k,k]$ homotopy-Padé technique not only can accelerate the convergence of the series solution but also exclude the auxiliary parameter $\hbar$ from it. This mainly becuase the Padé technique plays the role of a filter which filters out the most slowly decaying factors so as to accelerate the transient process and makes it stable, as shown by Cheng et al. [34].

From the numerical results in Figs. 2 and 3, for small values of $\hbar$, and the results in Fig. 4, it is clear that the approximate solutions obtained using the operator $\mathscr{L} = \frac{d}{dt}$ are better than approximate solutions obtained using the operator $\mathscr{L} = \frac{d^{\alpha}}{dt^{\alpha}}$ and this clearly indicates that the homotopy analysis method presented in this paper is a powerful method for handling

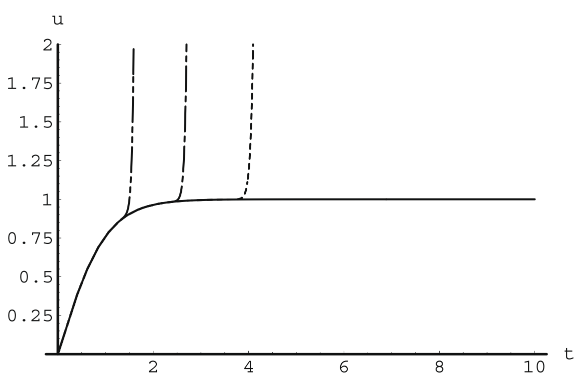

Fig. 1. Plots of the approximate solutions for Eq. (24) when $\alpha = 1$: (-··-) $\hbar = -1$; (-·-·) $\hbar = -0.5$; (····) $\hbar = -0.25$; (——) exact sol.

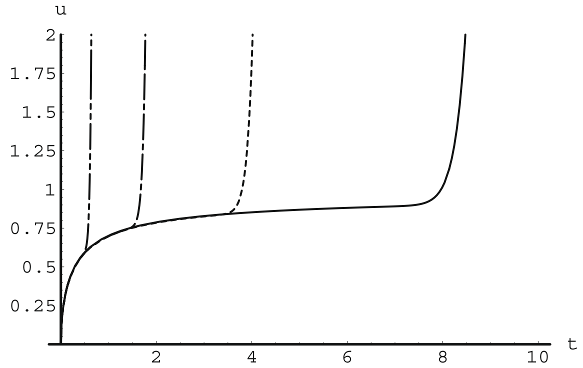

Fig. 2. Plots of the approximate solutions for Eq. (24) when $\alpha = 0.5$ and $\mathscr{L} = \frac{d^{0.5}}{dt^{0.5}}$: (-··-) $\hbar = -1$; (-·-·) $\hbar = -0.5$; (····) $\hbar = -0.25$; (——) $\hbar = -0.125$.

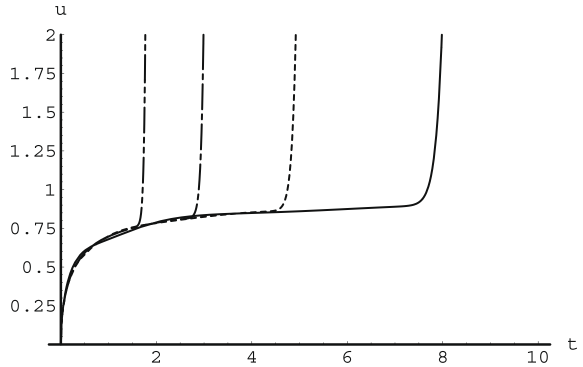

Fig. 3. Plots of the approximate solutions for Eq. (24) when $\alpha = 0.5$ and $\mathscr{L} = \frac{d}{dt}$: (-··-) $\hbar = -1$; (-·-·) $\hbar = -0.5$; (····) $\hbar = -0.25$; (——) $\hbar = -0.125$.

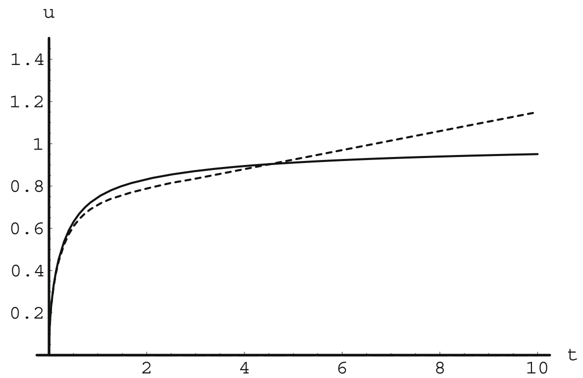

Fig. 4. [2,2] homotopy-Padé approximate solutions for Eq. (24) when $\alpha = 0.5$: (····) $\mathscr{L} = \frac{d^{0.5}}{dt^{0.5}}$; (——) $\mathscr{L} = \frac{d}{dt}$.

nonlinear fractional differential equations. It is to be noted that only 20 terms of the HAM series solution were used in evaluating the approximate solutions given in Figs. 1–3. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of $u(t)$.

Example 2. Consider the fractional differential equation

$$D_*^{\alpha} u + u' + u^3 = 0, \quad t > 0, \tag{34}$$

where $1 < \alpha \leqslant 2$, subject to the initial conditions

$$u(0) = 1, \quad u'(0) = 0. \tag{35}$$

In view of our algorithm, we can select the auxiliary linear operator as $\mathscr{L} = \frac{d^{\alpha}}{dt^{\alpha}}$ or $\mathscr{L} = \frac{d^2}{dt^2}$ and we can use the homotopy,

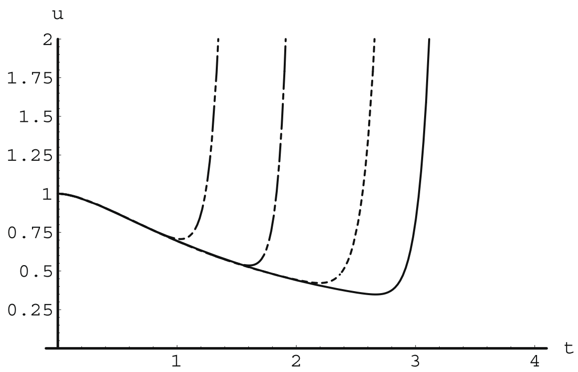

Fig. 5. Plots of the approximate solutions for Eq. (34) when $\alpha = 1.75$: (-···-) $\hbar = -1$ and $\mathscr{L} = \frac{d^{1.75}}{dt^{1.75}}$; (-·-·) $\hbar = -1$ and $\mathscr{L} = \frac{d^2}{dt^2}$; (····) $\hbar = -0.5$ and $\mathscr{L} = \frac{d^{1.75}}{dt^{1.75}}$; (——) $\hbar = -0.5$ and $\mathscr{L} = \frac{d^2}{dt^2}$.

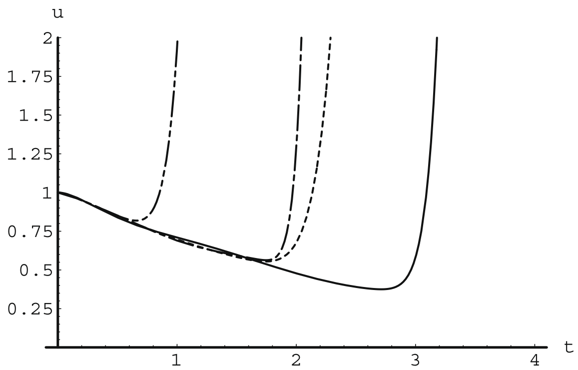

Fig. 6. Plots of the approximate solutions for Eq. (34) when $\alpha = 1.5$: (-···-) $\hbar = -1$ and $\mathscr{L} = \frac{d^{1.5}}{dt^{1.5}}$; (-·-·) $\hbar = -1$ and $\mathscr{L} = \frac{d^2}{dt^2}$; (····) $\hbar = -0.5$ and $\mathscr{L} = \frac{d^{1.5}}{dt^{1.5}}$; (——) $\hbar = -0.5$ and $\mathscr{L} = \frac{d^2}{dt^2}$.

$$(1-q)\mathscr{L}[\phi(t,q) - \phi_0(t)] = q\hbar H\big(D_*^{\alpha}\phi(t,q) + \phi'(t,q) + \phi^3(t,q)\big). \tag{36}$$

According to (23), we have the initial guess,

$$\phi_0 = 1. \tag{37}$$

Taking $H(t) = 1$ and following the same steps as given in the previous example, we can obtain two approximate solutions for Eq. (34).

Figs. 5 and 6 show the approximate solutions for Eq. (34) obtained using the operators $\mathscr{L} = \frac{d^{\alpha}}{dt^{\alpha}}$ and $\mathscr{L} = \frac{d^2}{dt^2}$ for different values of $\hbar$ when $\alpha = 1.75$ and $\alpha = 1.5$, respectively. As observed in the previous example the approximate solutions obtained using the operator $\mathscr{L} = \frac{d^2}{dt^2}$ are better than approximate solutions obtained using the operator $\mathscr{L} = \frac{d^{\alpha}}{dt^{\alpha}}$. A comparison between the results presented in Figs. 5 and 6 indicates that the modified homotopy analysis method presented in this paper can handle nonlinear differential equations of fractional order. It is to be noted that only 10 terms of the HAM series solution were used in evaluating the approximate solutions given in Figs. 5 and 6 using Mathematica software.

5. Concluding remarks

In this paper, we carefully proposed a reliable modification of the homotopy analysis method which introduces a promising tool for solving nonlinear differential equations of fractional order. Different from all other analytic methods, it provides us with a simple way to adjust and control the convergence region of solution series by introducing an auxiliary parameter $\hbar$. This work illustrates the validity and great potential of the homotopy analysis method for nonlinear fractional differential equations. The basic ideas of this approach can be further employed to solve other strongly nonlinear problems in fractional calculus.

In this paper, we showed the procedure of constructing the set of base functions and the high-order deformation equation in a simple form. Besides, we simply choose the classical integer operator as the auxiliary linear operator. In this way, we obtain approximate solutions in power series and these approximate solutions are in good agreement with the solutions obtained in [21] using fractional operators.

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