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A novel method for nonlinear fractional partial differential equations: Combination of DTM and generalized Taylor's formula

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Abstract

In this article, a novel numerical method is proposed for nonlinear partial differential equations with space- and time-fractional derivatives. This method is based on the two-dimensional differential transform method (DTM) and generalized Taylor's formula. The fractional derivatives are considered in the Caputo sense. Several illustrative examples are given to demonstrate the effectiveness of the present method. Results obtained using the scheme presented here agree well with the analytical solutions and the numerical results presented elsewhere. Results also show that the numerical scheme is very effective and convenient for solving nonlinear partial differential equations of fractional order.

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1. Introduction

Fractional order partial differential equations, as generalizations of classical integer order partial differential equations, are increasingly used to model problems in fluid flow, finance and other areas of application. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals prove to be more useful for the formulation of certain electrochemical problems than the classical models [26]. Fractional differentiation and integration operators are also used for extensions of the diffusion and wave equations [27].

A great deal of effort has been expended over the last 10 years or so in attempting to find robust and stable numerical and analytical methods for solving fractional partial differential equations of physical interest. Numerical and analytical methods have included finite difference method [11,29,9], Adomian decomposition method (ADM) [12–16,1,21], variational iteration method (VIM) [8,17,22,18], and homotopy perturbation method (HPM) [8,19,23]. The VIM and the ADM have been extensively used to solve fractional partial differential equations, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization.

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The present authors have written a series of papers solving linear partial differential equations and nonlinear partial differential equations of fractional order (see [12–19,1,21–23]. Recently they developed a semi-numerical method for solving linear partial differential equations of fractional order [24]. This method is named as generalized differential transform method (GDTM) and is based on the two-dimensional differential transform method (DTM) [31,4,7] and generalized Taylor's formula [25]. The present paper may be regarded as an extension of the latter paper [24] on nonlinear partial differential with space- and time-fractional derivatives of the form

$$\frac{\partial^{\mu} u}{\partial t^{\mu}} = \frac{\partial^{v} u}{\partial x^{v}} + N_f(u(x,t)), \quad m-1<\mu\leqslant m, \quad n-1<v\leqslant n, \quad n,m\in N, \tag{1.1}$$

where μ and v are parameters describing the order of the fractional time- and space-derivatives in the Caputo sense, respectively, and N_f is a nonlinear operator which might include other fractional derivatives with respect to the variables x and t. The function $u(x,t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t<0$ and $x<0$. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses.

There are several definitions of a fractional derivative of order $\alpha>0$ [26,5]. The two most commonly used definitions are the Riemann–Liouville and Caputo. The Riemann–Liouville fractional integration of order v is defined as

$$J_a^v f(x) = \frac{1}{\Gamma(v)} \int_a^x (x-t)^{v-1} f(t)\,\mathrm{d}t, \quad v>0, \quad x>0. \tag{1.2}$$

The next two equations define Riemann–Liouville and Caputo fractional derivatives of order v, respectively,

$$D_a^v f(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}(J_a^{m-v} f(x)), \tag{1.3}$$

$$D_{*a}^v f(x) = J_a^{m-v}\left(\frac{\mathrm{d}^m}{\mathrm{d}x^m} f(x)\right), \tag{1.4}$$

where $m-1<v\leqslant m$ and $m\in N$. For now, the Caputo fractional derivative will be denoted by D_*^v to maintain a clear distinction with the Riemann–Liouville fractional derivative.

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider the one-dimensional space- and time-fractional nonlinear partial differential equation (2.1), where the unknown function $u=u(x,t)$ is a assumed to be a causal function of space and time, respectively, and the fractional derivatives are taken in Caputo sense as follows:

Definition 1.1. For m to be the smallest integer that exceeds μ, the Caputo time-fractional derivative operator of order $\mu>0$ is defined as

$$D_{*t}^{\mu} u(x,t) = \frac{\partial^{\mu} u(x,t)}{\partial t^{\mu}} = \begin{cases} \dfrac{1}{\Gamma(m-\mu)} \int_0^t (t-\tau)^{m-\mu-1} \dfrac{\partial^m u(x,\tau)}{\partial \tau^m}\,\mathrm{d}\tau & \text{for } m-1<\mu<m, \\ \dfrac{\partial^m u(x,t)}{\partial t^m} & \text{for } \mu=m\in\mathbb{N} \end{cases} \tag{1.5}$$

and the space-fractional derivative operator of order $v>0$ is defined as

$$D_{*x}^{v} u(x,t) = \frac{\partial^{v} u(x,t)}{\partial x^{v}} = \begin{cases} \dfrac{1}{\Gamma(m-v)} \int_0^x (x-\theta)^{m-v-1} \dfrac{\partial^m u(\theta,t)}{\partial \theta^m}\,\mathrm{d}\theta & \text{for } m-1<v<m, \\ \dfrac{\partial^m u(x,t)}{\partial x^m} & \text{for } v=m\in\mathbb{N}. \end{cases} \tag{1.6}$$

For mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

2. Generalized Taylor's formula

In this section we present the generalized Taylor's formula that involves Caputo fractional derivatives. This generalization is presented in [25].

Generalized Taylor's formula: Suppose $(D_{*a}^{\alpha})^k f(x) \in C(a,b]$ for $k = 0, 1, \ldots, n+1$, where $0 < \alpha \leqslant 1$, then we have

$$f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}((D_{*a}^{\alpha})^{i} f)(a+) + \frac{((D_{*a}^{\alpha})^{n+1} f)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha} \tag{2.1}$$

with $a \leqslant \xi \leqslant x$, for each $x \in (a,b]$ and D_{*a}^{α} is the Caputo fractional derivative of order α, where $(D_{*a}^{\alpha})^k = D_{*a}^{\alpha} D_{*a}^{\alpha} \cdots D_{*a}^{\alpha}$. In case of $\alpha = 1$, the generalized Taylor's formula (2.1) reduces to the classical Talyor's formula.

Theorem 2.1. *Suppose that $(D_{*a}^{\alpha})^k f(x) \in C(a,b]$ for $k = 0, 1, \ldots, N+1$, where $0 < \alpha \leqslant 1$. If $x \in [a,b]$, then* [25]

$$f(x) \simeq \sum_{i=0}^{N} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}((D_{*a}^{\alpha})^{i} f)(a+). \tag{2.2}$$

Furthermore, there is a value ξ with $a \leqslant \xi \leqslant x$ so that the error term $R_N^{\alpha}(x)$ has the form

$$R_N^{\alpha}(x) = \frac{((D_{*a}^{\alpha})^{N+1} f)(\xi)}{\Gamma((N+1)\alpha+1)}(x-a)^{(N+1)\alpha}. \tag{2.3}$$

The accuracy of $R_N^{\alpha}(x)$ increases when we choose large N and decreases as the value of x moves away from the center a. Hence, we must choose N large enough so that the error does not exceed a specified bound. In the following theorem, we find precise conditions under which the exponents hold for arbitrary fractional operators. This result is very useful on our approach for solving differential equations of fractional order.

Theorem 2.2. *Suppose that $f(x) = x^{\lambda} g(x)$, where $\lambda > -1$ and $g(x)$ has the generalized power series expansion $g(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n\alpha}$ with radius of convergence $R > 0$, where $0 < \alpha \leqslant 1$. Then* [20]

$$D_{*a}^{\gamma} D_{*a}^{\beta} f(x) = D_{*a}^{\gamma+\beta} f(x) \tag{2.4}$$

for all $x \in (0, R)$ if one of the following conditions is satisfied:

(a) $\beta < \lambda + 1$ *and* γ *arbitrary*,
(b) $\beta \geqslant \lambda + 1$, γ *arbitrary, and* $a_k = 0$ *for* $k = 0, 1, \ldots, m-1$, *where* $m - 1 < \beta \leqslant m$.

The proof of Theorem 2.1 is given in [20] and the proof of Theorem 2.2 is given in [25].

3. Generalized two-dimensional DTM

The DTM was first applied in the engineering domain by [31]. In general, the DTM is applied to the solution of electric circuit problems. The DTM is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the DTM obtains a polynomial series solution by means of an iterative procedure. The method is well addressed in [20–22].

In this section we shall derive the generalized two-dimensional DTM that we have developed for the numerical solution of linear partial differential equations with space- and time-fractional derivatives [24]. The proposed method is based on the combination of the classical two-dimensional DTM [21,22] and generalized Taylor's formula [25].

Consider a function of two variables $u(x,y)$, and suppose that it can be represented as a product of two single-variable functions, i.e., $u(x,y) = f(x)g(y)$. Based on the properties of generalized two-dimensional differential transform [4,7], the function $u(x, y)$ can be represented as

$$u(x, y) = \sum_{k=0}^{\infty} F_{\alpha}(k)(x - x_0)^{k\alpha} \sum_{h=0}^{\infty} G_{\beta}(h)(y - y_0)^{h\beta}$$
$$= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k, h)(x - x_0)^{k\alpha}(y - y_0)^{h\beta}, \tag{3.1}$$

where $0 < \alpha, \beta \leqslant 1$, $U_{\alpha,\beta}(k, h) = F_{\alpha}(k)G_{\beta}(h)$ is called the spectrum of $u(x, y)$. The generalized two-dimensional differential transform of the function $u(x, y)$ is given by

$$U_{\alpha,\beta}(k, h) = \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)}[(D^{\alpha}_{*x_0})^k (D^{\beta}_{*y_0})^h u(x, y)]_{(x_0,y_0)}, \tag{3.2}$$

where $(D^{\alpha}_{*x_0})^k = D^{\alpha}_{*x_0} D^{\alpha}_{*x_0} \cdots D^{\alpha}_{*x_0}$, k-times. In this paper, the lowercase $u(x, y)$ represents the original function while the uppercase $U_{\alpha,\beta}(k, h)$ stands for the transformed function. In case of $\alpha = 1$ and $\beta = 1$ the generalized two-dimensional differential transform (3.1) reduces to the classical two-dimensional differential transform. Based on Definitions 3.1 and 3.2, we have the following results.

Theorem 3.1. *Suppose that $U_{\alpha,\beta}(k, h)$, $V_{\alpha,\beta}(k, h)$ and $W_{\alpha,\beta}(k, h)$ are the differential transformations of the functions $u(x, y)$, $v(x, y)$ and $w(x, y)$, respectively, then*

(a) *if $u(x, y) = v(x, y) \pm w(x, y)$, then $U_{\alpha,\beta}(k, h) = V_{\alpha,\beta}(k, h) \pm W_{\alpha,\beta}(k, h)$,*
(b) *if $u(x, y) = av(x, y)$, $a \in \mathbf{R}$, then $U_{\alpha,\beta}(k, h) = aV_{\alpha,\beta}(k, h)$,*
(c) *if $u(x, y) = v(x, y)w(x, y)$, then $U_{\alpha,\beta}(k, h) = \sum_{r=0}^{k}\sum_{s=0}^{h} V_{\alpha,\beta}(r, h - s)W_{\alpha,\beta}(k - r, s)$,*
(d) *if $u(x, y) = (x - x_0)^{n\alpha}(y - y_0)^{m\beta}$, then $U_{\alpha,\beta}(k, h) = \delta(k - n)\delta(h - m)$.*

Theorem 3.2. *If $u(x, y) = D^{\alpha}_{*x_0} v(x, y)$, $0 < \alpha \leqslant 1$, then*

$$U_{\alpha,\beta}(k, h) = \frac{\Gamma(\alpha(k + 1) + 1)}{\Gamma(\alpha k + 1)} V_{\alpha,\beta}(k + 1, h). \tag{3.3}$$

Proof. From Definition 3.1 we have

$$U_{\alpha,\beta}(k, h) = \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)}[(D^{\alpha}_{*x_0})^k (D^{\beta}_{*y_0})^h D^{\alpha}_{*x_0} v(x, y)]_{(x_0,y_0)},$$
$$= \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)}[(D^{\alpha}_{*x_0})^{k+1} (D^{\beta}_{*y_0})^h v(x, y)]_{(x_0,y_0)},$$
$$= \frac{\Gamma(\alpha(k + 1) + 1)}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)\Gamma(\alpha(k + 1) + 1)}[(D^{\alpha}_{*x_0})^{k+1} (D^{\beta}_{*y_0})^h v(x, y)]_{(x_0,y_0)},$$
$$= \frac{\Gamma(\alpha(k + 1) + 1)}{\Gamma(\alpha k + 1)} V_{\alpha,\beta}(k + 1, h). \qquad \square$$

Theorem 3.3. *If $u(x, y) = f(x)g(y)$ and the function $f(x) = x^{\lambda}h(x)$, where $\lambda > -1$, $h(x)$ has the generalized Taylor series expansion $h(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{\alpha k}$, and*

(a) *$\beta < \lambda + 1$ and α arbitrary or*
(b) *$\beta \geqslant \lambda + 1$, α arbitrary and $a_n = 0$ for $n = 0, 1, \ldots, m - 1$, where $m - 1 < \beta \leqslant m$.*

Then the generalized differential transform (3.2) *becomes*

$$U_{\alpha,\beta}(k, h) = \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)}[D^{\alpha k}_{*x_0} (D^{\beta}_{*y_0})^h u(x, y)]_{(x_0,y_0)}. \tag{3.4}$$

Proof. The proof follows immediately from the fact that $D^{\gamma_1}_{*x_0} D^{\gamma_2}_{*x_0} f(x) = D^{\gamma_1+\gamma_2}_{*x_0} f(x)$, under the conditions given in Theorem 2.2. □

Theorem 3.4. *If* $u(x, y)=D^{\gamma}_{*x_0} v(x, y)$, $m-1<\gamma\leqslant m$ *and* $v(x, y)=f(x)g(y)$, *the function* $f(x)$ *satisfies the conditions given in Theorem* 2.2, *then*

$$U_{\alpha,\beta}(k, h) = \frac{\Gamma(\alpha k + \gamma + 1)}{\Gamma(\alpha k + 1)} V_{\alpha,\beta}(k + \gamma/\alpha, h). \tag{3.5}$$

Proof. Using Theorem 3.3, we have,

$$\begin{aligned} U_{\alpha,\beta}(k, h) &= \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)} [(D^{\alpha k}_{*x_0})(D^{\beta}_{*y_0})^h D^{\gamma}_{*x_0} v(x, y)]_{(x_0,y_0)}, \\ &= \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)} [(D^{\alpha k+\gamma}_{*x_0})(D^{\beta}_{*y_0})^h v(x, y)]_{(x_0,y_0)}, \\ &= \frac{\Gamma(\alpha k + \gamma + 1)}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)\Gamma(\alpha k + \gamma + 1)} [(D^{\alpha k+\gamma}_{*x_0})(D^{\beta}_{*y_0})^h v(x, y)]_{(x_0,y_0)}, \\ &= \frac{\Gamma(\alpha k + \gamma + 1)}{\Gamma(\alpha k + 1)} V_{\alpha,\beta}(k + \gamma/\alpha, h). \quad \square \end{aligned}$$

Now, if the function $u(x, y) = f(x)g(y)$, $f(x)$ and $g(y)$ satisfies the conditions given in Theorem 2.2, then the generalized differential transform (3.2) becomes

$$U_{\alpha,\beta}(k, h) = \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)} [(D^{\alpha k}_{*x_0})(D^{\beta h}_{*y_0}) u(x, y)]_{(x_0,y_0)}. \tag{3.6}$$

Therefore, in this case, if $u(x, y) = D^{\gamma}_{*x_0} D^{\mu}_{*y_0} v(x, y)$, where $m - 1 < \gamma \leqslant m$, $n - 1 < \mu \leqslant n$ and the functions $f(x)$ and $g(y)$ satisfies the conditions given in Theorem 2.2, then we have the following result:

$$U_{\alpha,\beta}(k, h) = \frac{\Gamma(\alpha k + \gamma + 1)}{\Gamma(\alpha k + 1)} \frac{\Gamma(\beta h + \mu + 1)}{\Gamma(\alpha k + 1)} V_{\alpha,\beta}(k + \gamma/\alpha, h + \mu/\beta). \tag{3.7}$$

4. Analysis of the method

In this section we shall use the analysis presented in the previous section to construct our numerical method for solving the following nonlinear partial differential equation with space- and time-fractional derivatives

$$\frac{\partial^{\mu} u}{\partial t^{\mu}} = \frac{\partial^{\nu} u}{\partial x^{\nu}} + N_f(u(x, t)), \quad m - 1 < \mu \leqslant m, \quad n - 1 < \nu \leqslant n, \quad n, m \in N, \tag{4.1}$$

where N_f is a nonlinear operator which might include other fractional derivatives with respect to the variables x and t.

First, if $0 < \mu \leqslant 1$ and $0 < \nu \leqslant 1$, we suppose that the solution of the nonlinear equation (4.1) can be written as a product of single-valued functions. In this case, selecting $\alpha = \mu$, $\beta = \nu$ and applying Theorem 2.1 to both sides of Eq. (4.1), it transforms to

$$\frac{\Gamma(\alpha(h + 1) + 1)}{\Gamma(\alpha h + 1)} U_{\alpha,\beta}(k, h + 1) = \frac{\Gamma(\beta(k + 1) + 1)}{\Gamma(\beta k + 1)} U_{\alpha,\beta}(k + 1, h) + F_{\alpha,\beta}(k, h), \tag{4.2}$$

where $F_{\alpha,\beta}(k, h)$ is the generalized differential transformation of $N_f(u(x, t))$.

Second, if $m - 1 < \mu = m_1/m_2 \leqslant m$ and $0 < \nu \leqslant 1$, we suppose that the solution the nonlinear equation (4.1) can be written as a product of single-valued functions $u(x, t) = v(x)w(t)$, where the function $w(t)$ satisfies the conditions given in Theorem 2.2. In this case, selecting $\alpha = 1/m_2$, $\beta = \nu$, and applying Theorem 2.3 to both sides of Eq. (4.1), it transforms to

$$\frac{\Gamma(\alpha(h + 1) + m_1)}{\Gamma(\alpha h + 1)} U_{\alpha,\beta}(k, h + m_1) = \frac{\Gamma(\beta(k + 1) + 1)}{\Gamma(\beta k + 1)} U_{\alpha,\beta}(k + 1, h) + F_{\alpha,\beta}(k, h). \tag{4.3}$$

Finally, if $m-1<\mu=m_1/m_2\leqslant m$ and $n-1<v=n_1/n_2\leqslant n$, we suppose that the solution of the nonlinear equation (4.1) can be written as a product of single-valued functions $u(x,t)=v(x)w(t)$, where the functions $v(x)$ and $w(t)$ satisfy the conditions given in Theorem 2.2. In this case, selecting $\alpha=1/m_2$, $\beta=1/n_2$, and applying Theorem 2.3 to both sides of Eq. (4.1), it transforms to

$$\frac{\Gamma(\alpha(h+1)+m_1)}{\Gamma(\alpha h+1)}U_{\alpha,\beta}(k,h+m_1)=\frac{\Gamma(\beta(k+1)+n_1)}{\Gamma(\beta k+1)}U_{\alpha,\beta}(k+n_1,h)+F_{\alpha,\beta}(k,h). \tag{4.4}$$

In all the above cases, the solution of the nonlinear space and time-fractional equation (4.1), using Definition 3.1, can be written as

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{\alpha,\beta}(k,h)x^{k\alpha}y^{h\beta}. \tag{4.5}$$

5. Applications and results

In this section we consider a few examples that demonstrate the performance and efficiency of the generalized differential transform method for solving nonlinear partial differential equations with time- or space-fractional derivatives.

Example 5.1. Consider the following nonlinear time-fractional equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+u(x,t)u_x(x,t)=x+xt^2,\quad t>0,\quad 0<\alpha\leqslant 1, \tag{5.1}$$

subject to the initial condition

$$u(x,0)=0. \tag{5.2}$$

Suppose that the solution $u(x,t)$ can be represented as a product of single-valued functions, $u(x,t)=v(x)w(t)$. Selecting $\beta=1$ and applying Eq. (4.2), the recurrence relation for the time-fractional equation (5.1) is given by

$$\begin{aligned}U_{\alpha,1}(k,h+1)=\frac{\Gamma(\alpha h+1)}{\Gamma(\alpha(h+1)+1)}\Bigg\{&\delta(k-1)\delta(h)+\delta(k-1)\delta(h-2)\\ &-\sum_{r=0}^{k}\sum_{s=0}^{h}U_{\alpha,1}(r,h-s)(k-r+1)U_{\alpha,1}(k-r+1,s)\Bigg\}.\end{aligned} \tag{5.3}$$

The generalized two-dimensional differential transform of the initial condition (5.2) is

$$U_{\alpha,1}(k,0)=0. \tag{5.4}$$

Utilizing the recurrence relation (5.3) and the transformed initial condition (5.4), we get

$$\begin{aligned}U_{\alpha,1}(1,1)&=\frac{1}{\Gamma(\alpha+1)},\\ U_{\alpha,1}(1,3)&=\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left(1-\frac{1}{(\Gamma(\alpha+1))^2}\right),\\ U_{\alpha,1}(1,5)&=-\frac{2}{\Gamma(\alpha+1)}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\frac{\Gamma(4\alpha+1)}{\Gamma(4\alpha+1)}\left(1-\frac{1}{(\Gamma(\alpha+1))^2}\right),\end{aligned}$$

and the other coefficients equal zero for $k,h\leqslant 5$. Therefore, from (3.1), the approximate solution of the nonlinear equation (5.1) can be derived as

$$\begin{aligned}u(x,t)=x\Bigg[&\frac{1}{\Gamma(\alpha+1)}t^\alpha+\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left(1-\frac{1}{(\Gamma(\alpha+1))^2}\right)t^{3\alpha}\\ &-\frac{2}{\Gamma(\alpha+1)}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\frac{\Gamma(4\alpha+1)}{\Gamma(4\alpha+1)}\left(1-\frac{1}{(\Gamma(\alpha+1))^2}\right)t^{5\alpha}\Bigg].\end{aligned} \tag{5.5}$$

Table 1
Numerical values when $\alpha = 0.5$, 0.75 and 1.0 for Eq. (5.1)

t	x	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$		
		u_{GDTM}	u_{HPM}	u_{GDTM}	u_{HPM}	u_{GDTM}	u_{HPM}	u_{Exact}
0.2	0.25	0.123635	0.104573	0.080835	0.078306	0.050000	0.049989	0.050000
	0.50	0.247270	0.209146	0.161671	0.156612	0.100000	0.099978	0.100000
	0.75	0.370905	0.313720	0.242596	0.234919	0.1500010	0.149968	0.150000
	1.00	0.494540	0.418293	0.323342	0.313225	0.200000	0.199957	0.200000
0.4	0.25	0.177148	0.177229	0.135446	0.136806	0.100000	0.099645	0.100000
	0.50	0.354295	0.354458	0.270892	0.273612	0.200000	0.199290	0.200000
	0.75	0.531443	0.531686	0.406338	0.410418	0.300000	0.298935	0.300000
	1.00	0.70859	0.708915	0.541784	0.547225	0.400000	0.398580	0.400000
0.6	0.25	0.226965	0.230500	0.185529	0.185146	0.150000	0.147158	0.150000
	0.50	0.453931	0.461000	0.371057	0.370292	0.300800	0.294317	0.300000
	0.75	0.680896	0.691499	0.556586	0.555437	0.450000	0.441475	0.450000
	1.00	0.907861	0.921999	0.742114	0.740583	0.60000	0.588634	0.600000

Table 1 shows the approximate solutions for Eq. (5.1) obtained for different values of α using the GDTM, for $k, h \leqslant 5$. The values of $\alpha = 1$ is the only case for which we know the exact solution $u(x,t) = xt$ and our approximate solution using the GDTM is more accurate than the approximate solution obtained in [19] using the HPM.

Example 5.2. Consider the following nonlinear space-fractional Fisher's equation

$$\frac{\partial u}{\partial t} - \frac{\partial^{1.5} u}{\partial x^{1.5}} - u(x,t)(1 - u(x,t)) = x^2, \quad x > 0, \tag{5.6}$$

subject to the initial condition

$$u(x,0) = x. \tag{5.7}$$

The fractional space derivatives are used to model anomalous diffusion or dispersion, where a particle plume spreads at a rate inconsistent with the classical Brownian motion model, and the plume may be asymmetric. When a fractional derivative replaces the second derivative in a diffusion or dispersion model, it leads to enhanced diffusion (also called superdiffusion); see Refs. [6,3], and references therein.

Suppose that the solution $u(x,t)$ can be represented as a product of single-valued functions, $u(x,t) = v(x)w(t)$ where the function $v(x)$ satisfies the conditions given in Theorem 2.2. Selecting $\alpha = 1$, $\beta = 0.5$, and applying Eq. (4.3), the recurrence relation for the space-fractional Fisher's equation (5.6) is given by

$$\begin{aligned} U_{1,1/2}(k, h+1) = \frac{1}{h+1} \Bigg\{ & \frac{\Gamma(k/2+2)}{\Gamma(k/2+1)} U_{1,1/2}(k+3, h) \\ & + \sum_{r=0}^{k} \sum_{s=0}^{h} U_{1,1/2}(r, h-s)(\delta(k-r)\delta(s) - U_{1,1/2}(k-r, s)) \Bigg\} + \delta(h)\delta(k-4). \end{aligned} \tag{5.8}$$

The generalized two-dimensional differential transform of the initial condition (5.7) is

$$U_{1,1/2}(k,0) = \delta(k-2). \tag{5.9}$$

Utilizing the recurrence relation (5.8) and the transformed initial condition (5.9), the first few components of $U_{1,1/2}(k,h)$ are calculated and given in Table 2.

Table 2
The first components of $U_{1,1/2}(k,h)$ for Eq. (5.6)

	$U_{1,1/2}(k,0)$	$U_{1,1/2}(k,1)$	$U_{1,1/2}(k,2)$	$U_{1,1/2}(k,3)$	$U_{1,1/2}(k,4)$
$U_{1,1/2}(0,h)$	0	0	0	0	0
$U_{1,1/2}(1,h)$	0	0	0	$-\frac{\Gamma(5/2)}{3\Gamma(3/2)}$	$-\frac{\Gamma(5/2)}{3\Gamma(3/2)}$
$U_{1,1/2}(2,h)$	1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$
$U_{1,1/2}(3,h)$	0	0	0	0	$\frac{\Gamma(5/2)}{6\Gamma(3/2)}$
$U_{1,1/2}(4,h)$	0	0	-1	-1	$-\frac{7}{12}$

Table 3
Numerical values for Eq. (5.6)

x	$t=0.1$		$t=0.2$		$t=0.3$		$t=0.4$	
	u_{GDTM}	u_{VIM}	u_{GDTM}	u_{VIM}	u_{GDTM}	u_{VIM}	u_{GDTM}	u_{VIM}
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.110233	0.110401	0.120145	0.121559	0.128281	0.133294	0.132819	0.145276
0.2	0.220348	0.220589	0.240212	0.242240	0.257431	0.264612	0.269327	0.287146
0.3	0.330259	0.330565	0.359453	0.362026	0.384716	0.393833	0.402530	0.425163
0.4	0.439957	0.440329	0.477796	0.480928	0.509872	0.520995	0.531749	0.559433
0.5	0.549441	0.549880	0.595214	0.598938	0.632794	0.646078	0.656718	0.689927
0.6	0.658707	0.659214	0.711692	0.716018	0.753428	0.768969	0.777297	0.816392
0.7	0.767755	0.768320	0.827221	0.832097	0.871744	0.889430	0.893402	0.938285
0.8	0.876585	0.877185	0.941796	0.947058	0.987718	1.007079	1.004978	1.054714
0.9	0.985196	0.985786	1.055412	1.060735	1.101336	1.121360	1.111987	1.164378
1.0	1.093587	1.094096	1.168067	1.172904	1.212587	1.231524	1.214400	1.265508

Therefore, the approximate solution, for $k,h\leqslant 4$ of the nonlinear space-fractional Fisher's equation (5.6) can be derived as

$$u(x,t)=-\frac{\Gamma(5/2)}{3\Gamma(3/2)}(t^3+t^4)x^{1/2}+\left(1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4\right)x + \frac{\Gamma(5/2)}{6\Gamma(3/2)}t^4x^{3/2}-\left(t^2+t^3+\frac{7}{12}t^4\right)x^2. \tag{5.10}$$

In order to make comparison with VIM, we follow the analysis given in [23] and solve the nonlinear space-fractional equation (5.6) by using VIM to obtain the following fourth-order approximate solution:

$$u(x,t)=x+xt+(x-2x^2)\frac{t^2}{2}+\left(\frac{1}{2}x-3x^2+2x^3-\frac{2}{\Gamma(3/2)}x^{1/2}\right)\frac{t^3}{3} -\left(\frac{4}{3}x^2-\frac{8}{3}x^3+\frac{2}{3\Gamma(3/2)}x^{1/2}\right)\frac{t^4}{4}-\left(\frac{1}{4}x^2-\frac{5}{3}x^3+x^4\right)\frac{t^5}{5} +\left(\frac{1}{3}x^3-\frac{2}{3}x^4\right)\frac{t^6}{6}-\frac{x^4}{9}\frac{t^7}{7}. \tag{5.11}$$

Table 3 shows the approximate solutions for Eq. (5.6) using the GDTM, for $k,h\leqslant 4$, and the VIM. From the numerical results in Table 3, it is to conclude that the approximate solution obtained using the GDTM is in good agreement with the approximate solution obtained using the VIM for all values of x and t.

Example 5.3. Consider the following nonlinear space- and time-fractional hyperbolic equation

$$\frac{\partial^{\gamma} u}{\partial t^{\gamma}}=\frac{\partial}{\partial x}\left(u(x,t)\frac{\partial^{\beta} u}{\partial x^{\beta}}\right),\quad t>0,\quad x>0, \tag{5.12}$$

where $0<\beta\leqslant 1$ and $1<\gamma=m_1/m_2\leqslant 2$, subject to the initial conditions

$$u(x,0)=x^{2\beta},\quad u_t(x,0)=-2x^{2\beta}. \tag{5.13}$$

The space–time-fractional diffusion equation (5.12), in which the dispersive flux is described by a fractional space derivative, has been applied to modeling the anomalous or super diffusion of solutes observed in heterogeneous porous media [10,2]. Eq. (5.12) is a time-fractional version of the advection dispersion equation solved in [30] via particle tracking methods. A similar time-fractional equation was applied to mobile–immobile flow in [28], and the time derivative is shown to model power law retention times in the immobile zone.

Suppose that the solution $u(x,t)$ can be represented as a product of single-valued functions, $u(x,t)=v(x)w(t)$ where the function $w(t)$ satisfies the conditions given in Theorem 2.2. According to Eq. (4.3), selecting $\alpha=1/m_2$, the recurrence relation for the space–time and fractional hyperbolic equation (5.12) is given by

$$\begin{aligned}&\frac{\Gamma(\alpha h+\gamma+1)}{\Gamma(\alpha h+1)}U_{1/2,\beta}(k,h+m_1)\\&\quad=(k+1)\sum_{r=0}^{k+1}\sum_{s=0}^{h}\frac{\Gamma(\beta(k-r+2)+1)}{\Gamma(\beta(k-r+1)+1)}U_{1/2,\beta}(r,h-s)U_{1/2,\beta}(k-r+2,s).\end{aligned} \tag{5.14}$$

In case of $\gamma=2$ and $m_1=2$, the generalized two-dimensional DTM of the initial conditions (5.13) are

$$U_{1/2,\beta}(k,0)=\delta(k-2), \tag{5.15}$$

$$U_{1/2,\beta}(k,1)=-2\delta(k-2). \tag{5.16}$$

Utilizing the recurrence relation (5.14) and the transformed initial condition (5.15) and (5.16), we get

$$\begin{aligned}&U_{1/2,\beta}(2,0)=1,\\&U_{1/2,\beta}(2,1)=-2,\\&U_{1/2,\beta}(2,2)=\frac{3}{2}\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)},\\&U_{1/2,\beta}(2,3)=-2\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)},\\&U_{1/2,\beta}(2,4)=\frac{\Gamma(2\beta+1)}{4\Gamma(\beta+1)}\left(\frac{3\Gamma(2\beta+1)}{\Gamma(\beta+1)}+4\right),\\&U_{1/2,\beta}(2,5)=\frac{-3}{2}\frac{\Gamma(2\beta+1)^2}{\Gamma(\beta+1)^2}\end{aligned}$$

and the other coefficients equal zero for $k,h\leqslant 5$. Therefore, from (3.1), the approximate solution of the nonlinear space and time fractional hyperbolic equation (5.12) can be derived as

$$\begin{aligned}u(x,t)=&\left[1-2t+\frac{3\Gamma(2\beta+1)}{2\Gamma(\beta+1)}t^2-2\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)}t^3\right.\\&\left.+\frac{\Gamma(2\beta+1)}{4\Gamma(\beta+1)}\left(\frac{3\Gamma(2\beta+1)}{\Gamma(\beta+1)}+4\right)t^4-\frac{3\Gamma(2\beta+1)^2}{2\Gamma(\beta+1)^2}t^5\right]x^{2\beta}.\end{aligned} \tag{5.17}$$

Table 4 shows the approximate solutions for Eq. (5.12) obtained for $\alpha=2$ and $\beta=1$ using the GDTM, for $k,h\leqslant 20$. Eq. (5.12), for $\beta=1$, is solved in [22] using the VIM and ADM. The values of $\alpha=2$ and $\beta=1$ is the only case for which

Table 4
Numerical values when $\alpha = 2$ and $\beta = 1$ for Eq. (5.12)

t	x	u_{GDTM}	u_{ADM}	u_{VIM}	u_{Exact}
0.2	0.25	0.043403	0.0433951	0.043400	0.043403
	0.50	0.173611	0.173580	0.173600	0.173611
	0.75	0.390625	0.390556	0.390600	0.390625
	1.00	0.694444	0.694321	0.694460	0.694444
0.4	0.25	0.031888	0.031567	0.031779	0.031888
	0.50	0.127551	0.126268	0.127118	0.127551
	0.75	0.286990	0.284103	0.286015	0.286990
	1.00	0.510204	0.505072	0.508471	0.508471
0.6	0.25	0.024433	0.022005	0.023665	0.024414
	0.50	0.097730	0.088018	0.094660	0.097656
	0.75	0.219893	0.198040	0.212984	0.219727
	1.00	0.390921	0.352071	0.378638	0.390625

we know the exact solution $u(x,t) = (x/(t+1))^2$ and our approximate solution using the GDTM is high agreement with the approximate solution obtained in [22] using the VIM and ADM.

6. Conclusions

In this research, a new numerical method to tackle the nonlinear partial differential equations with space- and time-fractional derivatives is proposed. The new method is based on the two-dimensional DTM and generalized Taylor's formula. Comparison of the results obtained by using the GDTM with that obtained by other existing methods reveals that the present method is very effective and convenient for solving nonlinear partial differential equations of fractional order.

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